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Partial differential equations

A remark on the fractional Hardy inequality with a remainder term

Une remarque sur l'inégalité de Hardy fractionnaire avec reste

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ABSTRACT

We prove in this note the following sharpened fractional Hardy inequality:
Let $N \geqslant 1$, $0 < s < 1$, $N > 2s$, *and* $\Omega \subset \mathbb{R}^N$ *a bounded domain. Then for all* $1 < q < 2$, *there exists a positive constant* $C = C(\Omega, q, N, s)$ *such that for all* $u \in \mathcal{C}_0^\infty(\Omega)$,

$$a_{N,s} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \Lambda_{N,s} \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^{2s}}\,dx \geqslant C(\Omega, q, N, s) \int_{\Omega}\int_{\Omega} \frac{(u(x)-u(y))^2}{|x-y|^{N+qs}}\,dx\,dy, \tag{1}$$

where

$$a_{N,s} = 2^{2s-1}\pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|} \quad \text{and} \quad \Lambda_{N,s} = 2^{2s} \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}.$$

RÉSUMÉ

Dans cette note, nous proposons l'amélioration suivante de l'inégalité de Hardy fractionnaire :
Soient $N \geqslant 1$, $0 < s < 1$, $N > 2s$, *et* $\Omega \subset \mathbb{R}^N$ *un domaine borné. Alors, pour tout* $1 < q < 2$, *il existe une constante positive* $C \equiv C(\Omega, q, N, s)$ *telle que, pour tout* $u \in \mathcal{C}_0^\infty(\Omega)$,

$$a_{N,s} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \Lambda_{N,s} \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^{2s}}\,dx \geqslant C(\Omega, q, N, s) \int_{\Omega}\int_{\Omega} \frac{(u(x)-u(y))^2}{|x-y|^{N+qs}}\,dx\,dy,$$

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avec

$$a_{N,s}=2^{2s-1}\pi^{-\frac{N}{2}}\frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}\quad \text{et}\quad \Lambda_{N,s}=2^{2s}\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}.$$

Version française abrégée

Soient $N\geqslant 1$, $0<s<1$ tels que $N>2s$. Dans [10], l'auteur prouve l'inégalité de Hardy fractionnaire suivante : pour tout $u\in\mathcal{C}_0^\infty(\Omega)$,

$$\Lambda_{N,s}\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^{2s}}\,\mathrm{d}x\leqslant\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}|^2\,\mathrm{d}\xi,\quad \text{avec } \Lambda_{N,s}=2^{2s}\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}. \tag{2}$$

La constante $\Lambda_{N,s}$ est optimale et elle n'est pas atteinte.

Une amélioration de (2) a été obtenue par Frank, Lieb et Seiringer [8]. En effet, en reprenant la même notation que celle utilisée dans [8], si l'on pose :

$$h_s(u):=\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}|^2\,\mathrm{d}\xi-\Lambda_{N,s}\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^{2s}}\,\mathrm{d}x,$$

alors pour tout $\gamma<2_s^*\equiv\frac{2N}{N-2s}$, il existe $C=C(\gamma,N,s)>0$ telle que si $\Omega\subset\mathbb{R}^N$ avec $|\Omega|<\infty$, on a

$$h_s(u)\geqslant C(\gamma,N,s)|\Omega|^{2(\frac{1}{\gamma}-\frac{1}{2_s^*})}\|u\|_\gamma^2,\quad \text{pour tout } u\in\mathcal{C}_0^\infty(\Omega).$$

Notre résultat principal est le suivant :

Théorème 0.1. *Soient $N\geqslant 1$, $0<s<1$ avec $N>2s$. On suppose que $\Omega\subset\mathbb{R}^N$ est un domaine borné, alors pour tout $1<q<2$, il existe une constante positive $C=C(\Omega,q,N,s)$ telle que pour tout $u\in\mathcal{C}_0^\infty(\Omega)$,*

$$a_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y-\Lambda_{N,s}\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^{2s}}\,\mathrm{d}x\geqslant C(\Omega,q,N,s)\int_\Omega\int_\Omega\frac{(u(x)-u(y))^2}{|x-y|^{N+qs}}\,\mathrm{d}x\,\mathrm{d}y.$$

1. Introduction and main results

Let $N\geqslant 1$, $0<s<1$ be such that $N>2s$. Herbst [10] proved the following fractional Hardy inequality,

$$\Lambda_{N,s}\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^{2s}}\,\mathrm{d}x\leqslant\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}|^2\,\mathrm{d}\xi,\quad u\in\mathcal{C}_0^\infty(\mathbb{R}^N),\quad \text{where } \Lambda_{N,s}=2^{2s}\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}. \tag{3}$$

The constant $\Lambda_{N,s}$ is optimal and it is not achieved.

An improvement of inequality (3) has been obtained by Frank, Lieb, and Seiringer [8]. Indeed, following the same notation as in [8], we define

$$h_s(u):=\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}|^2\,\mathrm{d}\xi-\Lambda_{N,s}\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^{2s}}\,\mathrm{d}x. \tag{4}$$

Then for all $\gamma<2_s^*\equiv\frac{2N}{N-2s}$, there exists a universal constant $C=C(\gamma,N,s)>0$ such that for all $\Omega\subset\mathbb{R}^N$ with $|\Omega|<\infty$, the following inequality holds:

$$h_s(u)\geqslant C(\gamma,N,s)|\Omega|^{2(\frac{1}{\gamma}-\frac{1}{2_s^*})}\|u\|_\gamma^2,\quad \text{for all } u\in\mathcal{C}_0^\infty(\Omega). \tag{5}$$

Notice that inequality (5) shows in particular that the constant $\Lambda_{N,s}$ in (3) is not attained. An extension of (5) to the fractional Hardy inequality for the norm $p\neq 2$ has been obtained in [9] (see also [4,7] for related results).

Moreover, in the classical local framework, the following improved Hardy inequality was obtained in [11],

$$\int_\Omega |\nabla u|^2\,dx - \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2(x)}{|x|^2}\,dx \geqslant C(\Omega)\int_\Omega |\nabla u|^2\left(\log\left(\frac{R}{|x|}\right)\right)^{-2}dx, \tag{6}$$

where R is such that $\overline{\Omega} \subset B_R(0)$ and $C(\Omega) > 0$. See [1] for a different and simple proof.

In the fractional case, Fall [5] obtains a partial extension of this kind of estimate with a q-norm, $q < 2$, of a fractional gradient. He uses the harmonic extension to the positive upper space and the classical Dirichlet–Neumann transformation. See [3].

Let us recall the two following identities obtained in [8] (Lemma 3.1 and Proposition 4.1 respectively),

$$\int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}|^2\,d\xi = a_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy, \quad a_{N,s} = 2^{2s-1}\pi^{-\frac{N}{2}}\frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|} \tag{7}$$

and the *ground-state representation*, that is, calling $v(x) = u(x)|x|^{\frac{N-2s}{2}}$, then

$$h_s(u) = a_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v(x)-v(y))^2}{|x-y|^{N+2s}}\frac{dx}{|x|^{\frac{N-2s}{2}}}\frac{dy}{|y|^{\frac{N-2s}{2}}}. \tag{8}$$

In this note we obtain a Hardy inequality with a remainder term that is a 2-norm of a *fractional gradient*. Instead of the harmonic extension, we use the *ground-state representation* (8).

Precisely we formulate the main result as follows.

Theorem 1.1. *Let $N \geqslant 1$, $0 < s < 1$ and $N > 2s$. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain, then for all $1 \leqslant q < 2$, there exists a positive constant $C = C(\Omega, q, N, s)$ such that for all $u \in \mathcal{C}_0^\infty(\Omega)$,*

$$a_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \Lambda_{N,s}\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^{2s}}\,dx \geqslant C(\Omega,q,N,s)\int_\Omega\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+qs}}\,dx\,dy. \tag{9}$$

Remark 1. The right-hand side in (9) is exactly the norm in the space $H_0^\tau(\Omega)$ with $\tau = \frac{qs}{2}$. In particular, if we define $\tilde{H}(\Omega)$ as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $h_s(u)$, then $\tilde{H}(\Omega)$ is compactly embedded in $L^\gamma(\Omega)$ for all $\gamma < 2_s^*$. As a consequence, if $\gamma < 2_s^*$, and we consider the problem

$$\begin{cases} (-\Delta)^s u - \Lambda_{N,s}\dfrac{u}{|x|^{2s}} = |u|^{\gamma-2}u, \quad u > 0 \text{ in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \text{ where } (-\Delta)^s u =: a_{N,s} P.V. \displaystyle\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy, \end{cases} \tag{10}$$

then, there exists a *mountain pass solution* in $\tilde{H}(\Omega)$ (see [2] to complete details).

2. Proof of Theorem 1.1

For simplicity of typing, we set $\alpha = \frac{N-2s}{2}$, then $w(x) = |x|^{-\alpha}$ and $v(x) = \dfrac{u(x)}{w(x)}$. Thus,

$$\frac{(v(x)-v(y))^2}{|x-y|^{N+2s}}w(x)w(y) = \frac{((u(x)-u(y)) - \frac{u(y)}{w(y)}(w(x)-w(y)))^2}{|x-y|^{N+2s}}\frac{w(y)}{w(x)} \equiv f_1(x,y).$$

In the same way, thanks to the symmetry of $f_1(x,y)$, it immediately follows that

$$\frac{(v(x)-v(y))^2}{|x-y|^{N+2s}}w(x)w(y) = \frac{((u(y)-u(x)) - \frac{u(x)}{w(x)}(w(y)-w(x)))^2}{|x-y|^{N+2s}}\frac{w(x)}{w(y)} \equiv f_2(x,y).$$

Then,

$$h_s(u) = \frac{a_{N,s}}{2}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_1(x,y)\,dx\,dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_2(x,y)\,dx\,dy\right).$$

Since f_1 and f_2 are positive functions, it follows that

$$h_s(u) \geqslant \frac{a_{N,s}}{2}\left(\int_\Omega\int_\Omega f_1(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega\int_\Omega f_2(x,y)\,\mathrm{d}x\,\mathrm{d}y\right).$$

In what follows we will denote by C_i any positive constant that depends only on N, Ω, q and s and which can vary from line to line.

Notice that for all $(x,y) \in \Omega \times \Omega$ and $q < 2$, we have $\frac{1}{|x-y|^{N+2s}} \geqslant \frac{C}{|x-y|^{N+qs}}$ and

$$K(x,y) \equiv \frac{w(x)w(y)}{w^2(x)+w^2(y)} \leqslant \frac{1}{2}.$$

It is clear that $K(x,y)(\frac{w(x)}{w(y)} + \frac{w(y)}{w(x)}) = 1$. Hence,

$$f_1(x,y) \geqslant C_1 K(x,y)\frac{w(y)}{w(x)}\left[\frac{(u(x)-u(y))^2}{|x-y|^{N+qs}} + \left(\frac{u(y)}{w(y)}\right)^2\frac{(w(x)-w(y))^2}{|x-y|^{N+qs}} - 2\frac{u(y)}{w(y)}\frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+qs}}\right],$$

and

$$f_2(x,y) \geqslant C_1 K(x,y)\frac{w(x)}{w(y)}\left[\frac{(u(y)-u(x))^2}{|x-y|^{N+qs}} + \left(\frac{u(x)}{w(x)}\right)^2\frac{(w(y)-w(x))^2}{|x-y|^{N+qs}} - 2\frac{u(x)}{w(x)}\frac{(u(y)-u(x))(w(y)-w(x))}{|x-y|^{N+qs}}\right].$$

Therefore,

$$\begin{aligned} h_s(u) &\geqslant \frac{C_1}{2}\int_\Omega\int_\Omega K(x,y)\left(\frac{w(y)}{w(x)}+\frac{w(x)}{w(y)}\right)\frac{(u(x)-u(y))^2}{|x-y|^{N+qs}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - C_1\int_\Omega\int_\Omega K(x,y)\left[\frac{u(y)}{w(x)}\frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+qs}} + \frac{u(x)}{w(y)}\frac{(u(y)-u(x))(w(y)-w(x))}{|x-y|^{N+qs}}\right]\mathrm{d}x\,\mathrm{d}y \\ &\geqslant \frac{C_1}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+qs}}\,\mathrm{d}x\,\mathrm{d}y - C_1\int_\Omega\int_\Omega g_1(x,y)\,\mathrm{d}x\,\mathrm{d}y - C_1\int_\Omega\int_\Omega g_2(x,y)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

with

$$g_1(x,y) = K(x,y)\frac{u(y)}{w(x)}\left(\frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+qs}}\right)$$

and

$$g_2(x,y) = K(x,y)\frac{u(x)}{w(y)}\left(\frac{(u(y)-u(x))(w(y)-w(x))}{|x-y|^{N+qs}}\right).$$

Using the fact that $u \in \mathcal{C}_0^\infty(\Omega)$, it easily follows that $\int_\Omega\int_\Omega |g_i(x,y)|\,\mathrm{d}x\,\mathrm{d}y < \infty$ for $i = 1, 2$.

Since $g_2(x,y) = g_1(y,x)$, then we have just to estimate $\int_\Omega\int_\Omega g_2(x,y)\,\mathrm{d}x\,\mathrm{d}y$. Notice that

$$g_2(x,y) = \frac{w(x)u(x)}{w^2(x)+w^2(y)}\frac{(u(y)-u(x))(w(y)-w(x))}{|x-y|^{N+qs}}.$$

Hence, using Young's inequality, it follows

$$\int_\Omega\int_\Omega \left|g_2(x,y)\right|\mathrm{d}x\,\mathrm{d}y \leqslant \epsilon\int_\Omega\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+qs}}\,\mathrm{d}x\,\mathrm{d}y + C(\epsilon)\int_\Omega\int_\Omega \frac{w^2(x)u^2(x)(w(x)-w(y))^2}{(w^2(x)+w^2(y))^2|x-y|^{N+qs}}\,\mathrm{d}x\,\mathrm{d}y.$$

We claim that

$$I \equiv \int_\Omega\int_\Omega \frac{w^2(x)u^2(x)(w(x)-w(y))^2}{(w^2(x)+w^2(y))^2|x-y|^{N+qs}}\,\mathrm{d}x\,\mathrm{d}y \leqslant Ch_s(u).$$

It is clear that

$$I = \int_\Omega u^2(x)\left[\int_\Omega \frac{(|x|^\alpha-|y|^\alpha)^2|y|^{2\alpha}}{(|x|^{2\alpha}+|y|^{2\alpha})^2|x-y|^{N+qs}}\,\mathrm{d}y\right]\mathrm{d}x.$$

To compute the above integral, we follow closely the argument used in [6] (in the proof of Theorem 1.1). We set $y = \rho y'$ and $x = rx'$, then taking in consideration that $\Omega \subset B_R(0)$, we get:

$$I \leqslant \int_\Omega u^2(x) \int_0^R \frac{(r^\alpha - \rho^\alpha)^2 \rho^{N+2\alpha-1}}{(r^{2\alpha} + \rho^{2\alpha})^2} \left(\int_{|y'|=1} \frac{dH^{N-1}(y')}{|\rho y' - rx'|^{N+qs}} \right) d\rho\, dx.$$

Let $\rho = r\sigma$, then

$$I \leqslant \int_\Omega \frac{u^2(x)}{|x|^{qs}} \int_0^{\frac{R}{r}} \frac{(1-\sigma^\alpha)^2 \sigma^{N+2\alpha-1}}{(1+\sigma^{2\alpha})^2} \left(\int_{|y'|=1} \frac{dH^{N-1}(y')}{|\sigma y' - x'|^{N+qs}} \right) d\sigma\, dx \leqslant \int_\Omega \frac{u^2(x)}{|x|^{qs}}\, dx \int_0^\infty \frac{(1-\sigma^\alpha)^2 \sigma^{N+2\alpha-1}}{(1+\sigma^{2\alpha})^2} K(\sigma)\, d\sigma,$$

where

$$K(\sigma) = 2\frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \int_0^\pi \frac{\sin^{N-2}(\theta)}{(1 - 2\sigma\cos(\theta) + \sigma^2)^{\frac{N+qs}{2}}}\, d\theta.$$

Let analyze $\int_0^\infty \frac{(1-\sigma^\alpha)^2 \sigma^{N+2\alpha-1}}{(1+\sigma^{2\alpha})^2} K(\sigma)\, d\sigma$. We have:

$$\int_0^\infty \frac{(1-\sigma^\alpha)^2 \sigma^{N+2\alpha-1}}{(1+\sigma^{2\alpha})^2} K(\sigma)\, d\sigma = \int_0^1 \frac{(1-\sigma^\alpha)^2 \sigma^{N+2\alpha-1}}{(1+\sigma^{2\alpha})^2} K(\sigma)\, d\sigma + \int_1^\infty \frac{(1-\sigma^\alpha)^2 \sigma^{N+2\alpha-1}}{(1+\sigma^{2\alpha})^2} K(\sigma)\, d\sigma.$$

By setting $\xi = \frac{1}{\sigma}$ in the first integral and taking into account that $K(\frac{1}{\xi}) = \xi^{N+qs} K(\xi)$, we get:

$$\int_0^\infty \frac{(1-\sigma^\alpha)^2 \sigma^{N+2\alpha-1}}{(1+\sigma^{2\alpha})^2} K(\sigma)\, d\sigma = \int_1^\infty \frac{(\xi^\alpha - 1)^2 \xi^{qs-1}}{(1+\xi^{2\alpha})^2} K(\xi)\, d\xi + \int_1^\infty \frac{(\sigma^\alpha - 1)^2 \sigma^{N+2\alpha-1}}{(1+\sigma^{2\alpha})^2} K(\sigma)\, d\sigma.$$

Notice that $K(\sigma) \backsimeq \sigma^{-N-qs}$ as $\sigma \to \infty$, thus both integrals converge near ∞.

On the other hand and following the computation of [6] (estimates (2.1) to (2.10), see also the definition of the function H in formula (2.6)), we reach that $K(\sigma) \leqslant C(\sigma^2 - 1)^{-1-2s}$ as $\sigma \to 1$. Hence, as $\sigma, \xi \to 1^+$,

$$\left(\sigma^\alpha - 1\right)^2 K(\sigma) \leqslant C(\sigma - 1)^{1-2s} \in L^1(1,2).$$

Thus, combining the above estimates, we get $I \leqslant C \int_\Omega \frac{u^2(x)}{|x|^{qs}}\, dx$.

Since $q < 2$, using Hölder's inequality and (5), it follows that $I \leqslant C h_s(u)$. Then the claim follows.

As a conclusion, we obtain that

$$h_s(u) \geqslant C(\Omega, q, N, s) \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x-y|^{N+qs}}\, dx\, dy,$$

which concludes the proof.

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