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Differential geometry

The GBC mass for asymptotically hyperbolic manifolds ☆

La masse de Gauss–Bonnet–Chern sur des variétés asymptotiquement hyperboliques

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ABSTRACT

By using the Gauss–Bonnet curvature, we introduce a higher-order mass, the Gauss–Bonnet–Chern mass, for asymptotically hyperbolic manifolds and show that it is a geometric invariant. Moreover, we prove a positive mass theorem for this new mass for asymptotically hyperbolic graphs. Then, we prove the weighted Alexandrov–Fenchel inequalities in the hyperbolic space $\mathbb{H}^n$ for any horospherical convex hypersurface Σ. As an application, we obtain an optimal Penrose-type inequality for this new mass for asymptotically hyperbolic graphs with a horizon type boundary Σ, provided that a dominant energy condition $\widetilde{L}_k \geqslant 0$ holds.

RÉSUMÉ

En utilisant la courbure de Gauss–Bonnet, on introduit une nouvelle masse d'ordre supérieur – la masse de Gauss–Bonnet–Chern –, sur des variétés asymptotiquement hyperboliques. On montre qu'il s'agit d'un invariant géométrique. On démontre également le théorème de masse positive sur des graphes sur l'espace hyperbolique $\mathbb{H}^n$ et des inégalités d'Alexandrov–Fenchel à poids dans $\mathbb{H}^n$ pour toute hypersurface convexe de type horosphérique. Ainsi, on obtient une inégalité de type Penrose optimale pour cette masse sur toute variété asymptotiquement hyperbolique qui est graphe sur $\mathbb{H}^n$ avec un horizon au bord, à condition que la condition d'énergie dominante $\widetilde{L}_k \geqslant 0$ soit satisfaite.

1. Introduction

The Riemannian positive mass theorem (PMT), "*Any asymptotically flat Riemannian manifold $\mathcal{M}^n$ with a suitable decay order and with nonnegative scalar curvature has the nonnegative ADM mass*", plays an important role in differential geometry. This theorem was first proved by Schoen and Yau [15] for manifolds of dimension $n \leqslant 7$ and later for spin manifolds by Witten [17] using spinors. A refinement of the PMT is the Riemannian Penrose inequality:

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$$m_1 = m_{\mathrm{ADM}} \geqslant \frac{1}{2}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}, \tag{1.1}$$

where m_{ADM} is the ADM mass of the asymptotically flat Riemannian manifold with a horizon Σ and $|\Sigma|$ denotes the area of Σ. (1.1), was proved by Huisken–Illmann [11] and Bray [1] for $n = 3$. Later, Bray and Lee [2] generalized Bray's proof to the case $n \leqslant 7$. Recently, Lam [12] gave an elegant proof of PMT and (1.1) in all dimensions for an asymptotically flat manifold that can be realized as a graph in $\mathbb{R}^{n+1}$.

The ADM mass, together with the positive mass theorem, was generalized to asymptotically hyperbolic manifolds in [3,16,19]. For this asymptotically hyperbolic mass, the corresponding Penrose conjecture is: "*For asymptotically hyperbolic manifold* $(\mathcal{M}^n, g)$ *with an outermost horizon* Σ, *its mass satisfies:*

$$m_1^{\mathbb{H}} = m^{\mathbb{H}} \geqslant \frac{1}{2}\left\{\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n}{n-1}}\right\}, \tag{1.2}$$

provided that the dominant energy condition:

$$R_g \geqslant -n(n-1), \tag{1.3}$$

holds". Here R_g denotes the scalar curvature of g. Recently, motivated by the work of Lam [12], Dahl, Gicquaud, and Sakovich [4], on the one hand, and de Lima and Girão [5], on the other hand, proved the Penrose inequality (1.2) for asymptotically hyperbolic graphs over $\mathbb{H}^n$ with the help of a weighted hyperbolic Minkowski inequality, or a weighted hyperbolic Alexandrov–Fenchel inequality:

$$\int_{\Sigma} VH\,\mathrm{d}\mu \geqslant (n-1)\omega_{n-1}\left\{\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n}{n-1}}\right\}, \tag{1.4}$$

if Σ is star-shaped and mean-convex (i.e. $H > 0$), which was proved by de Lima and Girão [5].

Recently motivated by the Gauss–Bonnet gravity, we have introduced the Gauss–Bonnet–Chern mass m_{GBC} for asymptotically flat manifolds by using the following Gauss–Bonnet curvature:

$$L_k := \frac{1}{2^k}\delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}}, \tag{1.5}$$

where $R_{ij}{}^{sl}$ is the Riemannian curvature tensor. One can check that L_1 is just the scalar curvature R. For general k, it is just the Euler integrand in Chern's proof of the Gauss–Bonnet–Chern theorem if $n = 2k$. See a survey of Zhang [18]. A systematic study of L_k was first given by Lovelock [13]. The Gauss–Bonnet–Chern mass m_{GBC} for the asymptotically flat manifolds is defined in [6] by:

$$m_k = m_{\mathrm{GBC}} = \frac{(n-2k)!}{2^{k-1}(n-1)!\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} P_{(k)}^{ijlm} \partial_m g_{jl} \nu_i \,\mathrm{d}\mu, \tag{1.6}$$

where ω_{n-1} is the volume of $(n-1)$-dimensional standard unit sphere and S_r is the Euclidean coordinate sphere, $\mathrm{d}\mu$ is the volume element on S_r induced by the Euclidean metric and ν is the outward unit normal to S_r in $\mathbb{R}^n$. Here the $(0,4)$-tensor $P_{(k)}$ is defined by:

$$P_{(k)}^{stlm} := \frac{1}{2^k}\delta^{i_1 i_2 \cdots i_{2k-3} i_{2k-2} st}_{j_1 j_2 \cdots j_{2k-3} j_{2k-2} j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-3} i_{2k-2}}{}^{j_{2k-3} j_{2k-2}} g^{j_{2k-1} l} g^{j_{2k} m}. \tag{1.7}$$

This $(0,4)$-tensor $P_{(k)}$ has a crucial property that it is divergence-free, which guarantees that the Gauss–Bonnet–Chern mass is well defined and is a geometric invariant in [6]. In [6] and [7], we prove a positive mass theorem in the case where $\mathcal{M}$ is an asymptotically flat graph over $\mathbb{R}^n$ or $\mathcal{M}$ is conformal to $\mathbb{R}^n$, respectively. For our mass m_{GBC}, a corresponding Penrose conjecture was proposed in [6]:

$$m_k = m_{\mathrm{GBC}} \geqslant \frac{1}{2^k}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2k}{n-1}}. \tag{1.8}$$

Moreover, we proved in [6] that this conjecture is true for asymptotically flat graphs over $\mathbb{R}^n \backslash \Omega$ by using classical Alexandrov–Fenchel inequalities.

2. Hyperbolic Gauss–Bonnet–Chern mass and its Penrose inequality

In the paper [8], motivated by our previous work, by using the Gauss–Bonnet curvature we introduce a higher-order mass for asymptotically hyperbolic manifolds, which is a generalization of the mass introduced by Wang [16] and Cruściel–Herzlich [3]. See also [9,14,19]. However, if we use directly the Gauss–Bonnet curvature L_k, we can only obtain a mass proportional to the usual hyperbolic mass, rather than a new one. In order to define a higher-order mass for asymptotically hyperbolic manifolds, the crucial observation is a slight modification of the Gauss–Bonnet curvature. More precisely, on a Riemannian manifold $(\mathcal{M}^n, g)$, we consider a modified Riemann curvature tensor:

$$\widetilde{\mathrm{Riem}}_{ijsl}(g) = \widetilde{R}_{ijsl}(g) := R_{ijsl}(g) + g_{is}g_{jl} - g_{il}g_{js} \tag{2.1}$$

and a new Gauss–Bonnet curvature with respect to this tensor $\widetilde{Riem}$:

$$\widetilde{L}_k := \frac{1}{2^k}\delta^{i_1 i_2\cdots i_{2k-1}i_{2k}}_{j_1 j_2\cdots j_{2k-1}j_{2k}} \widetilde{R}_{i_1 i_2}{}^{j_1 j_2}\cdots \widetilde{R}_{i_{2k-1}i_{2k}}{}^{j_{2k-1}j_{2k}} = \widetilde{R}_{stlm}\widetilde{P}^{stlm}_{(k)}, \tag{2.2}$$

where

$$\widetilde{P}^{stlm}_{(k)} := \frac{1}{2^k}\delta^{i_1 i_2\cdots i_{2k-3}i_{2k-2}st}_{j_1 j_2\cdots j_{2k-3}j_{2k-2}j_{2k-1}j_{2k}} \widetilde{R}_{i_1 i_2}{}^{j_1 j_2}\cdots \widetilde{R}_{i_{2k-3}i_{2k-2}}{}^{j_{2k-3}j_{2k-2}} g^{j_{2k-1}l} g^{j_{2k}m}. \tag{2.3}$$

The tensor $\widetilde{P}_{(k)}$ has also the crucial property of being divergence free, which enables us to define a new mass.

Let us assume now that $2 \leqslant k < \frac{n}{2}$. We first introduce a "higher-order" mass for asymptotically hyperbolic manifolds with slower decay.

Definition 2.1. Assume that $(\mathcal{M}^n, g)$ is an asymptotically hyperbolic manifold of decay order $\tau > \frac{n}{k+1}$ and for $V \in \mathbb{N}_b := \{V \in C^\infty(\mathbb{H}^n) \mid \mathrm{Hess}^b V = Vb\}$, $V\widetilde{L}_k$ is integrable on $(\mathcal{M}^n, g)$. We define the Gauss–Bonnet–Chern mass integral with respect to the diffeomorphism Φ by:

$$H^\Phi_k(V) = \lim_{r\to\infty}\int_{S_r} \big((V\bar{\nabla}_l e_{ij} - e_{ij}\bar{\nabla}_l V)\widetilde{P}^{mijl}_{(k)}\big)\nu_m \,\mathrm{d}\mu, \tag{2.4}$$

where $e_{ij} := ((\Phi^{-1})^* g)_{ij} - b_{ij}$ and $\bar{\nabla}$ denotes the covariant derivative with respect to the hyperbolic metric b.

This definition is motivated by the work of Chruściel and Herzlich [3]. See also [9,14,16,19].

Theorem 2.2. *Suppose that $(\mathcal{M}^n, g)$ is an asymptotically hyperbolic manifold of decay order $\tau > \frac{n}{k+1}$ and for $V \in \mathbb{N}_b$, $V\widetilde{L}_k$ is integrable on $(\mathcal{M}^n, g)$, then the mass functional $H^\Phi_k(V)$ is well defined and does not depend on the choice of the coordinates at infinity used in the definition.*

From the mass functional H^Φ_k on $\mathbb{N}_b$, we define a higher-order mass, the Gauss–Bonnet–Chern mass for asymptotically hyperbolic manifolds as follows:

$$m^{\mathbb{H}}_k := c(n,k) \inf_{\mathbb{N}_b \cap \{V>0, \eta(V,V)=1\}} H^\Phi_k(V), \tag{2.5}$$

where $c(n,k) = \frac{(n-2k)!}{2^{k-1}(n-1)!\omega_{n-1}}$ is the normalization constant given in (1.6) and η is a Lorentz inner product. One may assume that the infimum in (2.5) is achieved by:

$$V = V_{(0)} = \cosh r,$$

where r is the hyperbolic distance to a fixed point $x_0 \in \mathbb{H}^n$. Therefore, we fix $V = V_{(0)} = \cosh r$.

Theorem 2.3 *(Positive Mass Theorem). Let $(\mathcal{M}^n, g) = (\mathbb{H}^n, b + V^2 \mathrm{d}f \otimes \mathrm{d}f)$ be the graph of a smooth asymptotically hyperbolic function $f : \mathbb{H}^n \to \mathbb{R}$ which satisfies $V\widetilde{L}_k$ is integrable and the graph $(\mathcal{M}^n, g)$ is asymptotically hyperbolic of decay order $\tau > \frac{n}{k+1}$. Then we have:*

$$m^{\mathbb{H}}_k = c(n,k)\int_{\mathcal{M}^n} \frac{1}{2}\frac{V\widetilde{L}_k}{\sqrt{1+V^2|\bar{\nabla} f|^2}}\,\mathrm{d}V_g. \tag{2.6}$$

In particular, $\widetilde{L}_k \geqslant 0$ implies $m^{\mathbb{H}}_k \geqslant 0$.

The condition:

$$\widetilde{L}_k \geqslant 0, \tag{2.7}$$

is a dominant energy condition, like (1.3). Such a beautiful expression (2.6) was found first by Lam for the scalar curvature R for asymptotically flat graphs over $\mathbb{R}^n$, and was generalized for the Gauss–Bonnet curvature in [6]. Dahl, Gicquaud, and Sakovich [4] obtained this formula for $m_1^{\mathbb{H}}$ for asymptotically hyperbolic graphs in $\mathbb{H}^n$. See also the work of de Lima and Girão [5] and of Huang and Wu [10].

Furthermore, if the manifold is an asymptotically hyperbolic graph with a horizon boundary, we establish a relationship between our new mass and a weighted higher-order mean curvature, as follows.

Theorem 2.4. *Let Ω be a bounded open set in $\mathbb{H}^n$ with boundary $\Sigma = \partial\Omega$. Assume $(\mathcal{M}^n, g) = (\mathbb{H}^n \setminus \Omega, b + V^2 \mathrm{d}f \otimes \mathrm{d}f)$ is an asymptotically hyperbolic manifold with a horizon Σ (i.e. $\partial\mathcal{M} = \partial\Omega \subset \mathcal{M}$ is minimal) which satisfies that $V\widetilde{L}_k$ is integrable. Moreover, assume that each connected component of Σ is in a level set of f and $|\bar{\nabla} f(x)| \to \infty$ as $x \to \Sigma$. Then*:

$$m_k^{\mathbb{H}} = c(n,k)\left(\frac{1}{2}\int_{\mathcal{M}^n} \frac{V\widetilde{L}_k}{\sqrt{1+V^2|\bar{\nabla} f|^2}}\,\mathrm{d}V_g + \frac{(2k-1)!}{2}\int_{\Sigma} V\sigma_{2k-1}\,\mathrm{d}\mu\right),$$

where σ_k denotes k-th mean curvature of Σ induced by the hyperbolic metric b.

In order to obtain a Penrose-type inequality for the hyperbolic mass $m_k^{\mathbb{H}}$ for asymptotically hyperbolic graphs with a horizon, we need to establish a "weighted" hyperbolic Alexandrov–Fenchel inequality. A hypersurface in $\mathbb{H}^n$ is *horospherical convex* if all principal curvatures are larger than or equal to 1.

Theorem 2.5. *Let Σ be a horospherical convex hypersurface in the hyperbolic space $\mathbb{H}^n$. We have*:

$$\int_{\Sigma} V\sigma_{2k-1}\,\mathrm{d}\mu \geqslant C_{n-1}^{2k-1}\omega_{n-1}\left(\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n}{k(n-1)}} + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2k}{k(n-1)}}\right)^k. \tag{2.8}$$

Equality holds if and only if Σ is a centered geodesic sphere in $\mathbb{H}^n$.

When $k = 1$, inequality (2.8) is just (1.4), which was proved by de Lima and Girão in [5]. These inequalities have their own interest in integral geometry as well as in differential geometry.

As a consequence of Theorems 2.4 and 2.5, the Penrose inequality for the Gauss–Bonnet–Chern mass $m_k^{\mathbb{H}}$ for asymptotically hyperbolic graphs with horizon boundaries follows.

Theorem 2.6 *(Penrose Inequality). Let Ω be a bounded open set in $\mathbb{H}^n$ and $\Sigma = \partial\Omega$. Assume $(\mathcal{M}^n, g) = (\mathbb{H}^n \setminus \Omega, b + V^2 \mathrm{d}f \otimes \mathrm{d}f)$ is an asymptotically hyperbolic manifold with a horizon Σ which satisfies that $V\widetilde{L}_k$ is integrable. Moreover, suppose that each connected component of Σ is in a level set of f and $|\bar{\nabla} f(x)| \to \infty$ as $x \to \Sigma$. Assume that each connected component of Σ is horospherical convex, then*:

$$m_k^{\mathbb{H}} \geqslant \frac{1}{2^k}\left(\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n}{k(n-1)}} + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2k}{k(n-1)}}\right)^k, \tag{2.9}$$

provided that

$$\widetilde{L}_k \geqslant 0.$$

Moreover, equality is achieved by the anti-de Sitter Schwarzschild type metric:

$$g_{\text{adS-Sch}} = \left(1 + \rho^2 - \frac{2m}{\rho^{\frac{n}{k}-2}}\right)^{-1}\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\Theta^2, \tag{2.10}$$

which is a generalization of the ordinary one. Here $\rho = \sinh r$ and $\mathrm{d}\Theta^2$ is the round metric on $\mathbb{S}^{n-1}$.

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