C. R. Acad. Sci. Paris, Ser. I 352 (2014) 55–58

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Functional analysis/Probability theory

A refinement of the Brascamp–Lieb–Poincaré inequality in one dimension

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ARTICLE INFO

Article history:
Received 3 November 2013
Accepted after revision 20 November 2013
Available online 20 December 2013

Presented by Jean-Pierre Kahane

ABSTRACT

In this short note, we give a refinement of the Brascamp–Lieb inequality in the style of the Houdré–Kagan extension for the Poincaré inequality in one dimension. This is inspired by works by Helffer and by Ledoux.

RÉSUMÉ

Dans cette brève Note, on donne un raffinement de l'inégalité de Brascamp–Lieb [1] dans le style de l'extension de Houdré–Kagan [3] pour l'inégalité de Poincaré en une dimension. Cette Note est inspirée par les travaux de Helffer et de Ledoux.

1. The Brascamp–Lieb inequality

We take a convex potential $V:\mathbb{R}\to\mathbb{R}$ which is C^k with $k\geqslant 2$ and the measure $\mu(\mathrm{d}x)=\mathrm{e}^{-V(x)}\,\mathrm{d}x$ which we assume to be a probability measure on $\mathbb{R}$.

Theorem 1. *(See Brascamp and Lieb [1].) If $V''>0$, then for any C^2 compactly supported function f on the real line*:

$$\operatorname{Var}_\mu(\phi)\leqslant\int\frac{(f')^2}{V''}\,\mathrm{d}\mu. \tag{1.1}$$

One of the proofs is due to Helffer [2] and we sketch it here as it were the starting point of our approach.

Consider the operator L acting on C^2 functions, given by:

$$L=-D^2+V'D$$

with $D\phi=\phi'$. We denote $\langle\cdot,\cdot\rangle$ the $L^2(\mu)$ inner product and observe that:

$$\langle L\phi,\phi\rangle=\left\|\phi'\right\|^2.$$

In particular, L can be extended to an unbounded non-negative operator on $L^2(\mu)$. From this, we get:

$$\|L\phi\|^2=\langle DL\phi,D\phi\rangle \tag{1.2}$$

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[1] The author was partially supported by a grant of the Romanian National Authority for Scientific Research, CNCS – UEFISCDI, project number PN-II-RU-TE-2011-3-0259 and by European Union Marie Curie Action Grant PIRG.GA.2009.249200.

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2013.11.013

and then if we take f a C^2 compactly supported function such that $\int f \, d\mu = 0$ and replace $\phi = L^{-1} f$, then we get:

$$\mathrm{Var}_\mu(f) = \langle f', DL^{-1} f \rangle.$$

Now a simple calculation reveals that:

$$DL = \left(L + V''\right)D$$

and then $(L + V'')^{-1} D = DL^{-1}$ where the inverses are defined appropriately. Therefore, we get:

$$\mathrm{Var}_\mu(f) = \left\langle \left(L + V''\right)^{-1} f', f' \right\rangle. \tag{1.3}$$

Since L is a non-negative operator, $(L + V'')^{-1} \leqslant (V'')^{-1}$ and this implies (1.1).

2. Refinements in the case of $\mathbb{R}$

We start with (1.3) and iterate it. This is inspired from [4], but without any use of the semigroup theory.

We let D be the derivation operator and we denote $D^* = -D + V'$ the adjoint of D with respect to the inner product in $L^2(\mu)$. In the sequel, for a given function F, we are going to denote by F also the multiplication operator by F. The main commutation relations are the content of the following.

Proposition 2. *Let $\mathcal{A}$ denote the operator defined on smooth positive functions E given by*

$$\mathcal{A}(E)(x) = \frac{1}{4}\left(2E''(x) + 2V'(x)E'(x) - \frac{E'(x)^2}{E(x)} + 4E(x)V''(x)\right). \tag{2.1}$$

(1) *If E is a positive function, then*

$$DED^* = \mathcal{A}(E) + E^{1/2} D^* D E^{1/2}. \tag{2.2}$$

(2) *For a positive function E,*

$$\left(E + D^*D\right)^{-1} = E^{-1} - E^{-1} D^* \left(I + DE^{-1}D^*\right)^{-1} DE^{-1}. \tag{2.3}$$

(3) *If E is a positive function such that $1 + \mathcal{A}(E^{-1})$ is positive and $F = E(1 + \mathcal{A}(E^{-1}))$, then*

$$\left(I + DE^{-1}D^*\right)^{-1} = F^{-1}E - E^{1/2}F^{-1}D^*\left(I + DF^{-1}D^*\right)^{-1} DF^{-1}E^{1/2}. \tag{2.4}$$

Proof. (1) We want to find two functions F and G such that:

$$DED^* = F + GD^*DG.$$

For this, take a function ϕ and write:

$$\left(DE\left(-D + V'\right)\right)\phi = \left(EV'\right)'\phi + \left(-E' + EV'\right)\phi' - E\phi''$$

while

$$F\phi + G\left(-D + V'\right)DG\phi = \left(F - GG'' + GG'V'\right)\phi + \left(G^2V' - 2GG'\right)\phi' - G^2\phi''$$

therefore it suffices to choose G such that:

$$G^2 = E \quad \text{and} \quad F = GG'' - GG'V' + \left(EV'\right)'$$

which means $G = E^{1/2}$ and $F = \mathcal{A}(E)$.

(2) We have:

$$\begin{aligned}\left(E + D^*D\right)^{-1} &= E^{-1} - E^{-1/2}\left(I - \left(I + E^{-1/2}D^*DE^{-1/2}\right)^{-1}\right)E^{-1/2} \\ &= E^{-1} - E^{-1}D^*\left(I + DE^{-1}D^*\right)^{-1}DE^{-1}\end{aligned}$$

where we used the fact that for any operator T,

$$I - \left(I + T^*T\right)^{-1} = T^*\left(I - TT^*\right)^{-1}T.$$

(3) From (2.2), we know that $I + DE^{-1}D^* = I + \mathcal{A}(E^{-1}) + E^{-1/2}D^*DE^{-1/2} = FE^{-1} + E^{-1/2}D^*DE^{-1/2}$ and from (2.3),

$$\big(FE^{-1} + E^{-1/2}D^*DE^{-1/2}\big)^{-1} = E^{1/2}\big(F + D^*D\big)^{-1}E^{1/2} = F^{-1}E - E^{1/2}F^{-1}D^*\big(I + DF^{-1}D^*\big)^{-1}DF^{-1}E^{1/2}. \qquad \square$$

Now, let us get back to the fact that $L = D^*D$ and that (1.3) gives:

$$\mathrm{Var}_\mu(f) = \big\langle \big(V'' + D^*D\big)^{-1}f', f'\big\rangle.$$

From (2.3) with $E_1 = V''$, we obtain first that:

$$\mathrm{Var}_\mu(f) = \big\langle \big(V''\big)^{-1}f', f'\big\rangle - \big\langle \big(I + DE_1^{-1}D^*\big)^{-1}D\big[E_1^{-1}f'\big], D\big[E_1^{-1}f'\big]\big\rangle. \tag{2.5}$$

It is interesting to point out that this provides the case of the equality Brascamp–Lieb if $D[(V'')^{-1}f'] = 0$, which can be solved for $f = C_1V' + C_2$.

Now we want to continue the inequality in (2.5) by taking $E_1 = E$ and using (2.5) for the case where $E_2 = E_1(I + \mathcal{A}(E_1^{-1})) > 0$; thus we go on with:

$$\big(I + DE_1^{-1}D^*\big)^{-1} = E_2^{-1}E_1 - E_1^{1/2}E_2^{-1}D^*\big(I + DE_2^{-1}D^*\big)^{-1}DE_2^{-1}E_1^{1/2}.$$

Hence we can write by setting $f_1 = E_1^{-1}f'$ and $f_2 = E_1^{1/2}D[f_1]$

$$\mathrm{Var}_\mu(f) = \big\|E_1^{-1/2}f'\big\|^2 - \big\|E_2^{-1/2}f_2\big\|^2 + \big\langle \big(I + DE_2^{-1}D^*\big)^{-1}D\big[E_2^{-1}f_2\big], D\big[E_2^{-1}f_2\big]\big\rangle.$$

Using a similar argument, let $E_3 = E_2(1 + \mathcal{A}(E_2^{-1}))$ provided that E_3 is positive. Then we can continue with:

$$\big(I + DE_2^{-1}D^*\big)^{-1} = I + \mathcal{A}\big(E_2^{-1}\big) - E_2^{1/2}E_3^{-1}D^*\big(I + DE_3^{-1}D^*\big)^{-1}DE_3^{-1}E_2^{1/2}$$

and letting $f_3 = E_2^{1/2}D[f_2]$, we obtain:

$$\mathrm{Var}_\mu(f) = \big\|E_1^{-1/2}f'\big\|^2 - \big\|E_2^{-1/2}f_2\big\|^2 + \big\|E_3^{-1/2}f_3\big\|^2 - \big\langle \big(I + DE_3^{-1}D^*\big)^{-1}D\big[E_3^{-1}f_3\big], D\big[E_3^{-1}f_3\big]\big\rangle.$$

By induction, we can define:

$$E_1 = V'' \quad \text{and} \quad f_1 = E_1^{-1}f' \tag{2.6}$$

$$E_n = E_{n-1}\big(1 + \mathcal{A}\big(E_{n-1}^{-1}\big)\big) \quad \text{and} \quad f_n = E_{n-1}^{1/2}D[f_{n-1}]. \tag{2.7}$$

Notice that here E_n is defined only if E_{n-1} is defined and positive, and we will assume that the sequence is defined as long as this condition is satisfied. We get the following result.

Theorem 3. *If $E_1, E_2, \ldots, E_n$ are positive functions, then for any compactly supported function f,*

$$\begin{aligned}\mathrm{Var}_\mu(f) = {} & \big\|E_1^{-1/2}f'\big\|^2 - \big\|E_2^{-1/2}f_2\big\|^2 + \cdots + (-1)^{n-1}\big\|E_n^{-1/2}f_n\big\|^2 \\ & + (-1)^n\big\langle \big(I + DE_n^{-1}D^*\big)^{-1}D\big[E_n^{-1}f_n\big], D\big[E_n^{-1}f_n\big]\big\rangle.\end{aligned} \tag{2.8}$$

In particular, for n even,

$$\mathrm{Var}_\mu(f) \geqslant \big\|E_1^{-1/2}f'\big\|^2 - \big\|E_2^{-1/2}f_2\big\|^2 + \cdots + (-1)^{n-1}\big\|E_n^{-1/2}f_n\big\|^2$$

and for n odd,

$$\mathrm{Var}_\mu(f) \leqslant \big\|E_1^{-1/2}f'\big\|^2 - \big\|E_2^{-1/2}f_2\big\|^2 + \cdots + (-1)^{n-1}\big\|E_n^{-1/2}f_n\big\|^2.$$

For $V(x) = x^2/2 - \log(\sqrt{2\pi})$ this leads to the following version of Houdré–Kagan inequality [3] due to Ledoux [4].

Corollary 4. *For $V(x) = x^2/2 - \log(\sqrt{2\pi})$ and f which is C^n with compact support, the following holds true*:

$$\mathrm{Var}_\mu(f) = \big\|f'\big\|^2 - \frac{1}{2!}\big\|f''\big\|^2 + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\big\|f^{(n-1)}\big\|^2 + \frac{(-1)^n}{(n-1)!}\big\langle (n+L)^{-1}f^{(n)}, f^{(n)}\big\rangle.$$

Another particular case of Theorem 3 is the following reverse-type Brascamp–Lieb inequality.

Corollary 5. *Provided that* $1+\mathcal{A}((V'')^{-1})>0$*, the following holds*:

$$\operatorname{Var}_\mu(f)\geqslant\big\langle\big(V''\big)^{-1}f',f'\big\rangle-\big\langle\big(1+\mathcal{A}\big(\big(V''\big)^{-1}\big)\big)^{-1}D\big[\big(V''\big)^{-1}f'\big],D\big[\big(V''\big)^{-1}f'\big]\big\rangle.$$

Furthermore, $1+\mathcal{A}((V'')^{-1})>0$ *is equivalent to*

$$3V^{(3)}(x)^2+8V''(x)^3-2V^{(4)}(x)V''(x)-2V^{(3)}(x)V''(x)V'(x)>0. \tag{2.9}$$

For instance, in the case where $a,b>0$ and

$$V(x)=ax^2/2+bx^4/4+C$$

(where C is the normalizing constant which makes μ a probability), the condition (2.9) reads as:

$$2a^3-3ab+\big(15a^2b+18b^2\big)x^2+42ab^2x^4+45b^3x^6>0 \tag{$*$}$$

for any x. In particular, for $x=0$, this gives $3b<2a^2$, which turns out to be enough to guarantee ($*$) for any other x. For the next corrections, the condition $1+\mathcal{A}(E_2^{-1})>0$ becomes equivalent to:

$$\begin{aligned}&4a^9-18a^7b+27a^3b^3+\big(90a^8b-225a^6b^2+504a^4b^3+540a^2b^4\big)x^2\\&\quad+\big(916a^7b^2-756a^5b^3+4203a^3b^4-162ab^5\big)x^4\\&\quad+\big(5563a^6b^3+2172a^4b^4+11124a^2b^5+1944b^6\big)x^6+\big(22326a^5b^4+23868a^3b^5+7209ab^6\big)x^8\\&\quad+\big(61689a^4b^5+74817a^2b^6-5832b^7\big)x^{10}+\big(117864a^3b^6+109026ab^7\big)x^{12}+\big(150741a^2b^7+63180b^8\big)x^{14}\\&\quad+117450ab^8x^{16}+42525b^9x^{18}>0\end{aligned}$$

for all x. This turns out to be equivalent to $b<\frac{1}{3}(-1+\sqrt{3})a^2$. In general, for higher corrections, the condition $E_n>0$ appears to be equivalent to a condition of the form $b<a^2t_n$ for some $t_n>0$ that is decreasing in n to 0. We do not have a solid proof of this, but some numerical simulations suggest this conclusion.

Another example is the potential $V(x)=x^2/2-a\log(x^2)+C$ with $a>0$, for which condition (2.9) becomes equivalent to:

$$4a^3-3ax^2+12a^2x^2+7ax^4+x^6>0$$

for all x. This turns out to be equivalent to $a>a_0$, where a_0 is the solution in $(0,1)$ of the equation $108-855a+144a^2+272a^3=0$ and numerically is $a_0\approx0.129852$. For the second-order correction, a numerical simulation indicates that we need to take $a>a_1$ with $a_1\approx0.314584$. Some numerical approximations suggest that $E_n>0$ is equivalent to $a>a_n$ with a_n being an increasing sequence to infinity.

Acknowledgements

The author wants to thank Michel Ledoux for an interesting conversation on this subject and to the reviewer of this paper for comments which led to its improvement.

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