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Complex analysis

Pointwise estimate for the Bergman kernel of the weighted Bergman spaces with exponential type weights

Estimation ponctuelle du noyau de Bergman des espaces à poids de type exponentiel

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A R T I C L E   I N F O

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A B S T R A C T

Let $AL^2_\phi(\mathbb{D})$ denote the closed subspace of $L^2(\mathbb{D}, e^{-2\phi}\,d\lambda)$ consisting of holomorphic functions in the unit disc $\mathbb{D}$. For certain class of subharmonic functions $\phi : \mathbb{D} \to \mathbb{D}$, we prove an upper pointwise estimate for the Bergman kernel for $AL^2_\phi(\mathbb{D})$.

R É S U M É

Soit $AL^2_\phi(\mathbb{D})$ le sous-espace fermé de $L^2(\mathbb{D}, e^{-2\phi}\,d\lambda)$ formé des fonctions holomorphes sur le disque unité $\mathbb{D}$. Pour une classe de fonctions sous-harmoniques $\phi : \mathbb{D} \to \mathbb{D}$, on établit une estimation ponctuelle du noyau de Bergman de $AL^2_\phi(\mathbb{D})$.

1. Introduction and statement of the main result

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and $d\lambda$ be its Lebesgue measure. For a measurable function $\phi : \mathbb{D} \to \mathbb{D}$, let $L^2_\phi(\mathbb{D})$ be the Hilbert space of measurable function f on $\mathbb{D}$ such that:

$$\|f\|_{L^2_\phi} := \left(\int_{\mathbb{D}} |f|^2 e^{-2\phi}\, d\lambda \right)^{\frac{1}{2}} < \infty$$

Let $AL^2_\phi(\mathbb{D})$ be the closed subspace of $L^2_\phi(\mathbb{D})$ consisting of analytic functions. Let P be the orthogonal projection of $L^2_\phi(\mathbb{D})$ onto $AL^2_\phi(\mathbb{D})$:

$$Pf(z) := \int_{\mathbb{D}} K(z,w) f(w)\, e^{-2\phi(w)}\, d\lambda$$

where K is the reproducing kernel of P.

The purpose of this note is to give an upper pointwise estimate of K for some class of subharmonic functions ϕ on $\mathbb{D}$ introduced by Oleinik [10] and Oleinik and Perel'man [11].

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Definition 1.1. For $\phi \in C^2(\mathbb{D})$ and $\Delta\phi > 0$ put $\tau = (\Delta\phi)^{-1/2}$ where Δ is the Laplace operator. We call $\phi \in \mathcal{OP}(\mathbb{D})$ if the following conditions hold.

(1) $\exists C_1 > 0$ such that $|\tau(z) - \tau(w)| \leqslant C_1|z - w|$,
(2) $\exists C_2 > 0$ such that $\tau(z) \leqslant C_2(1 - |z|)$,
(3) $\exists 0 < C_3 < 1$ and $a > 0$ such that $\tau(w) \leqslant \tau(z) + C_3|z - w|$ for $w \notin D(z, a\tau(z))$ where $D(z, a\tau(z)) = \{w \in \mathbb{D},\ |w - z| \leqslant a\tau(z)\}$.

Some examples of functions in $\mathcal{OP}(\mathbb{D})$ are as follows:

(i) $\phi_1(z) = -\frac{A}{2}\log(1 - |z|^2)$, $A > 0$.
(ii) $\phi_2(z) = \frac{1}{2}(-A\log(1 - |z|^2) + B(1 - |z|^2)^{-\alpha})$, $A \geqslant 0$, $B > 0$, $\alpha > 0$.
(iii) $\phi_1 + h$ and $\phi_2 + h$ where ϕ_1 and ϕ_2 are as in (i) and (ii) respectively and $h \in C^2(\mathbb{D})$ can be any harmonic function on $\mathbb{D}$.

For $z, w \in \mathbb{D}$, the distance d_ϕ induced by the metric $\tau(z)^{-2}\,\mathrm{d}z \otimes \mathrm{d}\bar{z}$ is given by:

$$d_\phi(z, w) = \inf_\gamma \int_0^1 \frac{|\gamma'(t)|}{\tau(\gamma(t))}\,\mathrm{d}t$$

where γ runs over the piecewise C^1 curves $\gamma : [0, 1] \to \mathbb{D}$ with $\gamma(0) = z$ and $\gamma(1) = w$. Thanks to condition (2), the metric space $(\mathbb{D}, d_\phi)$ is complete and $d_\phi \succcurlyeq d_\mathrm{h}$ where d_h is the hyperbolic distance.

Our main result is the following theorem on the off-diagonal decay of the Bergman kernel.

Theorem 1.2. *Let $\phi \in \mathcal{OP}(\mathbb{D})$ and K be the Bergman kernel for $AL^2_\phi(\mathbb{D})$. There exist positive constants C and σ such that for any $z, w \in \mathbb{D}$:*

$$|K(z, w)|\mathrm{e}^{-(\phi(z)+\phi(w))} \leqslant C\frac{1}{\tau(z)\tau(w)}\exp(-\sigma d_\phi(z, w))$$

In [4] and [9], M. Christ and J. Marzo and J. Ortega-Cerdà obtained pointwise estimates for the Bergman kernel of the weighted Fock space $\mathcal{F}^2_\phi(\mathbb{C})$ under the hypothesis that $\Delta\phi$ is a doubling measure. This result was extended to several variables by H. Delin and H. Lindholm in [5] and [8] under a similar hypothesis.

In [12], A.P. Schuster and D. Varolin obtained a pointwise estimate for the Bergman kernel of the weighted Bergman space $AL^2(\mathbb{D}, \mathrm{e}^{-2\phi}(1 - |z|^2)^{-2}\,\mathrm{d}\lambda)$ under the hypothesis that $\Delta\phi$ is comparable to the hyperbolic metric of $\mathbb{D}$:

$$|K(z, w)|\mathrm{e}^{-(\phi(z)+\phi(w))} \leqslant C\exp(-\sigma d_\mathrm{h}(z, w))$$

For $\phi \in \mathcal{OP}(\mathbb{D})$ and under the strong condition: $\forall m \geqslant 1$, $\exists b_m > 0$ and $0 < t_m < \frac{1}{m}$ such that:

$$\tau(w) \leqslant \tau(z) + t_m|z - w| \quad \text{if } |z - w| > b_m\tau(z),$$

H. Arroussi and J. Pau [1] give the following pointwise estimate: for each $k \geqslant 1$, there exists $C_k > 0$ such that:

$$|K(z, w)|\mathrm{e}^{-(\phi(z)+\phi(w))} \leqslant \frac{C_k[d_\tau(z, w)]^{-k}}{\tau(z)\tau(w)}$$

where $d_\tau(w) = \frac{|z-w|}{\min[\tau(z),\tau(w)]}$. A better estimate will be:

$$|K(z, w)|\mathrm{e}^{-(\phi(z)+\phi(w))} \leqslant \frac{C}{\tau(z)\tau(w)}\mathrm{e}^{-\sigma d_\tau(z,w)}.$$

2. Proof of Theorem 2.1

Near the diagonal, by [7, Lemma 3.6] there exists $\alpha > 0$ sufficiently small such that:

$$|K(z, w)| \sim \sqrt{K(z, z)}\sqrt{K(w, w)} \sim \frac{\mathrm{e}^{\phi(z)+\phi(w)}}{\tau(z)\tau(w)} \quad \text{if } |z - w| \leqslant \alpha \min[\tau(z), \tau(w)]$$

Off the diagonal, let $|z - w| > \alpha \min[\tau(z), \tau(w)]$ and $\beta > 0$ such that $D(z, \beta\tau(z)) \cap D(w, \beta\tau(w)) = \emptyset$. We may suppose that $\tau(z) \leqslant \tau(w)$. Fix a smooth function $\chi \in C_0^\infty(\mathbb{D})$ such that $\operatorname{supp}\chi \subset D(w, \beta\tau(w))$, $0 \leqslant \chi \leqslant 1$, $\chi = 1$ in $D(w, \frac{\beta}{2}\tau(w))$ and $|\bar\partial\chi|^2 \preccurlyeq \chi\tau(w)^{-2}$. Since $\phi \in \mathcal{OP}(\mathbb{D})$, by [10, Lemmas 1 and 2], the following mean inequality holds:

$$|K(w, z)|^2\mathrm{e}^{-2\phi(w)} \preccurlyeq \frac{1}{\tau(w)^2}\int_{D(w,\frac{\beta}{2}\tau(w))} \chi(\zeta)|K(\zeta, z)|^2\mathrm{e}^{-2\phi(\zeta)}\,\mathrm{d}\lambda(\zeta) \preccurlyeq \frac{1}{\tau(w)^2}\|K(., z)\|^2_{L^2(\chi\mathrm{e}^{-2\phi}\,\mathrm{d}\lambda)} \qquad (*)$$

Hence $\|K(.,z)\|_{L^2(\chi e^{-\phi})} = \sup_f |\langle f, K(.,z)\rangle_{L^2(\chi \mathrm{e}^{-2\phi}\,\mathrm{d}\lambda)}|$, where f is holomorphic in $D(w, \beta\tau(w))$ with $\|f\|_{L^2(\chi \mathrm{e}^{-2\phi}\,\mathrm{d}\lambda)} = 1$. Since $P_\phi(f\chi)(z) = \langle f, K(.,z)\rangle_{L^2(\chi \mathrm{e}^{-2\phi}\,\mathrm{d}\lambda)}$ and that $u_f = f\chi - P_\phi(f\chi)$ is the minimal solution in $L^2(\mathbb{D}, \mathrm{e}^{-2\phi}\,\mathrm{d}\lambda)$ of $\bar\partial u = f\bar\partial\chi$, and from the fact that $\chi(z) = 0$, we have:

$$\left|\langle f, K(.,z)\rangle_{L^2(\chi \mathrm{e}^{-2\phi}\,\mathrm{d}\lambda)}\right| = \left|P_\phi(f\chi)(z)\right| = \left|u_f(z)\right|$$

Since $D(z, \beta\tau(z)) \cap D(w, \beta\tau(w)) = \emptyset$, the function u_f is holomorphic in $D(z, \nu\tau(z))$ for some $\nu > 0$. By the mean value inequality:

$$\left|u_f(z)\right|^2 \mathrm{e}^{-2\phi(z)} \preccurlyeq \frac{1}{\tau(z)^2} \int\limits_{D(z,\nu\tau(z))} \left|u_f(\zeta)\right|^2 \mathrm{e}^{-2\phi(\zeta)}\,\mathrm{d}\lambda \preccurlyeq \frac{1}{\tau(z)^2} \int\limits_{D(z,\nu\tau(z))} \mathrm{e}^{-\epsilon\frac{|\zeta - z|}{\nu\tau(z)}} \left|u_f(\zeta)\right|^2 \mathrm{e}^{-2\phi(\zeta)}\,\mathrm{d}\lambda$$

Since the linear curve $\gamma(t) = (1-t)z + t\zeta$ lies in $D(z, \nu\tau(z))$ and $\tau(\gamma(t)) \sim \tau(z)$, we have $d_\phi(\zeta, z) \leqslant C\frac{|\zeta - z|}{\tau(z)}$ for $\zeta \in D(z, \nu\tau(z))$. Hence

$$\left|u_f(z)\right|^2 \mathrm{e}^{-2\phi(z)} \preccurlyeq \frac{1}{\tau(z)^2} \int\limits_{D(z,\nu\tau(z))} \mathrm{e}^{-C\epsilon d_\phi(\zeta,z)} \left|u_f(\zeta)\right|^2 \mathrm{e}^{-2\phi(\zeta)}\,\mathrm{d}\lambda \preccurlyeq \frac{1}{\tau(z)^2} \int\limits_{\mathbb{D}} \mathrm{e}^{-C\epsilon d_\phi(\zeta,z)} \left|u_f(\zeta)\right|^2 \mathrm{e}^{-2\phi(\zeta)}\,\mathrm{d}\lambda$$

The function $\zeta \to d_\phi(\zeta, z)$ is smooth on $\mathbb{D} \setminus \mathrm{Cut}(z) \cup \{z\}$ where $\mathrm{Cut}(z)$ is the cut locus: the set of all cut points of z along all geodesics that start from z. To get a smooth Lipschitz approximation of d_ϕ, we recall the following result of Greene and Wu [6] (see also [2]).

Theorem 2.1. *Let M be a complete Riemannian manifold, let $h : M \to \mathbb{R}$ be a Lipschitz function, let $\eta : M \to]0, +\infty[$ be a continuous function, and r a positive number. Then there exists a C^∞ smooth Lipschitz function $g : M \to \mathbb{R}$ such that $|h(x) - g(x)| \leqslant \eta(x)$ for every $x \in M$, and $Lip(g) \leqslant Lip(h) + r$.*

We use this result with $h(\zeta) = d_\phi(\zeta, z)$, $\eta = 1$ and $r = 1$. We have $d_\phi(\zeta, z) \prec g_z(\zeta) \prec d_\phi(\zeta, z)$ and $\tau(\zeta)|\,\mathrm{d}g_z(\zeta)| \leqslant 2$. Hence

$$\left|u_f(z)\right|^2 \mathrm{e}^{-2\phi(z)} \preccurlyeq \frac{1}{\tau(z)^2} \int\limits_{\mathbb{D}} \mathrm{e}^{-C\epsilon g_z(\zeta)} \left|u_f(\zeta)\right|^2 \mathrm{e}^{-2\phi(\zeta)}\,\mathrm{d}\lambda$$

By Berndtsson–Delin's improved L^2 estimates of for the minimal solution of $\bar\partial$ in $L^2(\mathbb{D}, \mathrm{e}^{-2\phi}\,\mathrm{d}\lambda)$ [3,5], we have:

$$\int\limits_{\mathbb{D}} \mathrm{e}^{-C\epsilon g_z(\zeta)} \left|u_f(\zeta)\right|^2 \mathrm{e}^{-2\phi(\zeta)}\,\mathrm{d}\lambda \preccurlyeq \int\limits_{\mathbb{D}} \mathrm{e}^{-C\epsilon g_z(\zeta)} \left|\bar\partial\chi(\zeta)\right|^2 \left|f(\zeta)\right|^2 \tau(\zeta)^2 \mathrm{e}^{-2\phi(\zeta)}\,\mathrm{d}\lambda$$

provided that $\tau|\partial\omega_\epsilon| \leqslant \mu\omega_\epsilon$ with $\mu < \sqrt{2}$ where $\omega_\epsilon(\zeta) = \mathrm{e}^{-C\epsilon g_z(\zeta)}$. If we choose ϵ small enough so that $\mu = 2C\epsilon < \sqrt{2}$ then $\tau|\partial\omega_\epsilon| = C\epsilon\tau|\partial g_z|\omega_\epsilon \leqslant \mu\omega_\epsilon$. Thus

$$\left|u_f(z)\right|^2 \mathrm{e}^{-2\phi(z)} \preccurlyeq \frac{1}{\tau(z)^2} \int\limits_{D(w,\beta\tau(w))} \mathrm{e}^{-C\epsilon d_\phi(\zeta,z)} \chi(\zeta)|f|^2 \mathrm{e}^{-2\phi(\zeta)}\,\mathrm{d}\lambda$$

where for the last term we use $\tau(\zeta) \sim \tau(w)$. Since $\zeta \in D(w, \beta\tau(w))$, we have:

$$d_\phi(\zeta, z) \geqslant d_\phi(z, w) - d_\phi(w, \zeta) \succcurlyeq d_\phi(z, w) - \frac{|\zeta - w|}{\beta\tau(w)} \succcurlyeq d_\phi(z, w)$$

and thanks to $(*)$, we conclude:

$$\left|K(z, w)\right| \mathrm{e}^{-(\phi(w)+\phi(z))} \leqslant \frac{C}{\tau(z)\tau(w)} \mathrm{e}^{-\sigma d_\phi(z,w)}.$$

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