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Nonlinear reduced basis approximation of parameterized evolution equations via the method of freezing

Approximation non linéaire utilisant des bases réduites d'équations d'évolution paramétrées par une méthode de figeage

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ABSTRACT

We present a new method for the nonlinear approximation of the solution manifolds of parameterized nonlinear evolution problems, in particular in hyperbolic regimes with moving discontinuities. Given the action of a Lie group on the solution space, the original problem is reformulated as a partial differential algebraic equation system by decomposing the solution into a group component and a spatial shape component, imposing appropriate algebraic constraints on the decomposition. The system is then projected onto a reduced basis space. We show that efficient online evaluation of the scheme is possible and study a numerical example showing its strongly improved performance in comparison to a scheme without freezing.

RÉSUMÉ

On présente une nouvelle méthode d'approximation non linéaire des variétés de solutions de problèmes d'évolution non linéaires paramétrées, en particulier dans les régimes hyperboliques. Pour une action de groupe de Lie donnée sur l'espace des solutions, le problème initial est reformulé comme une équation aux derivées partielles algébriques, en décomposant la solution en une partie sur le groupe et une partie sous forme spatiale. On impose ensuite des contraintes algébriques sur la décomposition. Dans la suite, on projette le système sur un espace de base réduite. On démontre que la méthode peut être évaluée « en ligne » de manière efficace, et on traite un exemple numérique montrant une perfomance améliorée si on la compare à la même méthode sans figeage.

Version française abrégée

Les méthodes de bases réduites (BR) sont des outils efficaces pour approcher les variétés des solutions de problèmes d'évolution paramétrées par des espaces de petites dimensions. Pour les problèmes linéaires, l'algorithme de POD–Greedy [4] est établi dans une approche de la réduction. Récemment, on a montré que cet algorithme est optimal au sens où les taux de

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convergence exponentiels ou algébriques de la n^e largeur de Kolmogorov sont conservés [3]. Cette approche a été étendue aux problèmes non linéaires basés sur l'interpolation empirique d'opérateurs non linéaires [2].

Pour ce qui concerne les problèmes dans lesquels la convection à faible régularité est prépondérante, la performance de la méthode BR est pourtant limitée par le fait que la translation des fonctions est non linéaire par rapport au vecteur de translation : la solution doit être approche en tous les points de l'espace où elle est transportée, ce qui entraîne une diminution seulement linéaire de la n^e largeur. On cherche donc à étendre les approches mentionnées ci-dessus pour les utiliser dans le cas de transformations non linéaires des espaces de bases réduites données par une action de groupe adaptée à l'espace des solutions.

Si $u_\mu(t) \in V$ est la solution d'un problème de Cauchy non linéaire paramétré de la forme (1), la « méthode de figeage » permet d'obtenir une décomposition $u_\mu(t) = g_\mu(t) . v_\mu(t)$ de u_μ en une partie de groupe $g_\mu(t) \in G$ et une partie de forme $v_\mu(t)$ pour une action lisse quelconque de groupe de Lie, pourvu que la condition d'équivalence (4) soit satisfaite. Cette approche, à l'origine développée pour étudier des équilibres relatifs d'équations d'évolution [1,8], est appliquée à un système d'équations aux derivées partielles algébriques pour $v_\mu(t)$ et par une équation differentielle ordinaire pour $g_\mu(t)$. Ce système remplace l'équation originelle de départ. Pour $v_\mu(t)$, le système est donné par (5), où, pour les élements $\mathfrak{h}$ de l'algèbre de Lie LG de G, l'opérateur $\mathcal{L}^G_{\mu,\mathfrak{h}}$ est défini par $\mathcal{L}^G_{\mu,\mathfrak{h}}(u) = \mathcal{L}_\mu(u) + \mathfrak{h} . u$, Φ étant une contrainte algébrique appelée « condition de phase ». Cette contrainte compense les dim G degrés de liberté additionels introduits par la décomposition (2) et force un changement minimal de $v_\mu(t)$ en fonction du temps. Sous l'hypothèse de l'existence d'un produit scalaire sur V, un choix possible pour Φ est d'imposer que l'évolution de $v_\mu(t)$ soit orthogonale à l'action de LG, ce qui conduit à (6), où $\mathfrak{e}_r$ est une base de LG. La partie de groupe $g_\mu(t)$ peut être reconstruite à partir de $\mathfrak{g}_\mu(t)$ comme solution du problème de Cauchy $\partial_t g_\mu(t) = g_\mu(t) \cdot \mathfrak{g}_\mu(t)$, $g_\mu(0) = 1_G$.

En choisissant les discrétisations $\mathbb{L}_\mu$, $\mathbb{L}^G_{\mu,\mathfrak{h}}$, $\mathbb{S}^G_r$ pour les opérateurs $\mathcal{L}_\mu$, $\mathcal{L}^G_{\mu,\mathfrak{h}}$ et les actions des $\mathfrak{e}_r$, on obtient le schéma discret figé de la Définition 2.1. Étant donné un sous-espace reduit de petite dimension V_N de l'espace discret V_H, on projette ce schéma discret sur V_N en utilisant un opérateur de projection approché P_N. Pour réaliser une évaluation en ligne rapide des opérateurs projetés, on utilise une interpolation empirique des opérateurs [2] pour approcher les opérateurs non linéaires $\mathbb{L}_\mu$ et $\mathbb{L}^G_{\mu,\mathfrak{h}}$, ce qui conduit au schéma figé-BR donné par (8) et $v^0_{N,\mu} = P_N(v^0_\mu)$. La Proposition 3.1 démontre qu'une décomposition « hors ligne/en ligne » du schéma est efficacement possible.

Pour illustrer le potentiel de cette nouvelle méthode, on considère le problème de Burgers [5] donné par (9) dans le domaine $\Omega = [0,2] \times [0,1]$ avec des conditions aux bords périodiques. Le paramètre μ peut varier dans l'intervalle $[1,2]$; de plus, $\mathbf{b} = [\,1\ 1\,]^T$, $t \in [0, 0{,}3]$, et $u_0(x_1, x_2) = 1/2(1 + \sin(2\pi x_1)\sin(2\pi x_2))$. Si $G = \mathbb{R}^2$ agit par translation de Ω, le système figé est donné par (10). On choisit un schéma aux volumes finis sur une grille de 120×60 éléments et 100 pas de temps pour la discrétisation de (10). Pour déterminer l'espace réduit et les dates pour l'interpolation des opérateurs, on utilise les algorithmes de POD–Greedy et EI–Greedy de [2]. Dans la Fig. 1, pour $\mu = 1{,}5$, on compare l'approximation détaillée de u_μ (colonne de gauche) à l'approximation réduite de u_μ (colonne du centre) et à l'approximation réduite de la solution figée v_μ (colonne de droite), en utilisant chaque fois une base de 20 vecteurs. La Fig. 2 démontre que le schéma figé-BR améliore l'erreur d'approximation de 1,7 ordres de grandeur si on la compare au même schéma de réduction sans figeage.

1. Introduction

Reduced basis (RB) methods are effective tools for approximating the solution manifolds of parameterized evolution problems by low-dimensional linear spaces, enabling fast online evaluation of the solution for arbitrary parameter values. For linear problems, the POD–Greedy algorithm [4] is by now an established reduction approach that has recently been proved to be optimal in the sense that exponential or algebraic convergence rates of the Kolmogorov n-width are maintained by the algorithm [3]. The approach has been further extended to nonlinear settings in [2] based on empirical interpolation of the nonlinear operators. For convection dominated problems with low regularity, however, the performance of RB-methods is limited by the fact that the translation of functions is nonlinear in the translation vector: the solution has to be approximated at every location in space it is being transported to, resulting in merely linear decay of the n-width. Our aim is therefore to extend the above approaches by additionally allowing transformations of the reduced spaces given by an appropriate group action on the solution space. Originally developed for the study of relative equilibria of evolution equations [1,8], the *method of freezing* allows us to obtain such a decomposition of the solution into a group and shape component for arbitrary Lie group actions, provided they satisfy the equivariance condition (4). Combining this method with RB-techniques, we obtain a new nonlinear reduction method for parameterized nonlinear evolution equations. The method is evaluated numerically for a two-dimensional Burgers-type problem introduced in [5]. For further results on reduced basis model reduction for Burgers-type problems using the POD–Greedy approach, we refer to, e.g., [7,6] and the references therein.

2. The method of freezing

Assume we are given a parameter-dependent nonlinear Cauchy problem of the form:

$$\partial_t u_\mu(t) + \mathcal{L}_\mu\big(u_\mu(t)\big) = 0, \quad u_\mu(0) = u_0 \tag{1}$$

where $u_\mu(t) \in V$, $t \in [0, T]$ and $\mathcal{L}_\mu$ is a partial differential operator acting on the function space V. Given a Lie group G acting smoothly on V by linear operators, we want to decompose the solution $u_\mu(t)$ into a group component $g_\mu(t) \in G$ and a shape component $v_\mu(t) \in V$ such that:

$$u_\mu(t) = g_\mu(t) \,.\, v_\mu(t) \tag{2}$$

where the action of G on V is denoted by a lower dot. As a guiding example, consider the action of the Lie groups $G = \mathbb{R}^d$ on functions $v : \mathbb{R}^d \longrightarrow \mathbb{R}$ via translation, i.e. $(g \,.\, v)(x) = v(x - g)$ for $g \in \mathbb{R}^d$. In this case, if $u_\mu(t)$ is a solution of (1) drifting along a certain trajectory in space, we want to find a decomposition (2) such that the drift is captured by the evolution of $g_\mu(t)$, whereas $v_\mu(t)$ becomes as stationary as possible, only describing the change of the shape of $u_\mu(t)$ over time. Note that, in general, the group action does not have to be induced by a mapping of the underlying spatial domain (e.g., shifts or rotations), but can also involve more general transformations of V. Inserting (2) into (1), we formally get:

$$\partial_t g_\mu(t) \,.\, v_\mu(t) + g_\mu(t) \,.\, \partial_t v_\mu(t) + \mathcal{L}_\mu\big(g_\mu(t) \,.\, v_\mu(t)\big) = 0$$

which can be rewritten as:

$$\partial_t v_\mu(t) + g_\mu(t)^{-1} \,.\, \mathcal{L}_\mu\big(g_\mu(t) \,.\, v_\mu(t)\big) + \mathfrak{g}_\mu(t) \,.\, v_\mu(t) = 0, \quad \mathfrak{g}_\mu(t) = g_\mu(t)^{-1} \cdot \partial_t g_\mu(t) \tag{3}$$

where $\mathfrak{g}_\mu(t)$ is from the Lie algebra $\mathrm{L}G$ of G, i.e. the tangential space of G at its neutral element 1_G.

Since the decomposition (2) introduces $\dim G$ additional degrees of freedom, the resulting system is now underdetermined. It is the main idea of the method of freezing to compensate for these degrees of freedom by adding appropriate algebraic constraints Φ that force v_μ to have minimal change over time. These constraints are called *phase conditions*. Thus, the group component of the solution is automatically determined by the phase condition, and no a priori knowledge of the evolution of the solution is necessary. If one further assumes that the operator $\mathcal{L}_\mu$ is *equivariant* under the group action, i.e.:

$$g^{-1} \,.\, \mathcal{L}_\mu(g \,.\, v) = \mathcal{L}_\mu(v) \quad \forall g \in G,\ v \in V, \tag{4}$$

the system decouples into the partial differential algebraic equation (PDAE) of index one:

$$\partial_t v_\mu(t) + \mathcal{L}^G_{\mu,\mathfrak{g}_\mu(t)}\big(v_\mu(t)\big) = 0, \qquad \Phi\big(v_\mu(t), \mathfrak{g}_\mu(t)\big) = 0 \tag{5}$$

with $\mathcal{L}^G_{\mu,\mathfrak{h}}(u) = \mathcal{L}_\mu(u) + \mathfrak{h} \,.\, u$ and the ordinary differential equation $\partial_t g_\mu(t) = g_\mu(t) \cdot \mathfrak{g}_\mu(t)$, called the *reconstruction equation*. The initial conditions are $v_\mu(0) = u_0$ and $g_\mu(0) = 1_G$.

Different choices of phase conditions are possible [1]. We will restrict ourselves here to the so-called *orthogonality phase condition*: assuming that V is equipped with an inner product, we require that the evolution of v_μ is at each point in time orthogonal to the action of $\mathrm{L}G$, i.e. $(\partial_t v_\mu(t), \mathfrak{h} \,.\, v_\mu(t)) = 0$ for all $\mathfrak{h} \in \mathrm{L}G$. Inserting (5), we obtain:

$$\big(\mathcal{L}_\mu\big(v_\mu(t)\big), \mathfrak{h} \,.\, v_\mu(t)\big) + \big(\mathfrak{g}_\mu(t) \,.\, v_\mu(t), \mathfrak{h} \,.\, v_\mu(t)\big) = 0 \quad \forall \mathfrak{h} \in \mathrm{L}G.$$

After choosing a basis for $\mathrm{L}G$, this leads to a linear $\dim G \times \dim G$ equation system for $\mathfrak{g}_\mu(t)$. Denoting the basis vectors by $\mathfrak{e}_1, \ldots, \mathfrak{e}_{\dim G}$, this system can be written as:

$$\Phi\big(v_\mu(t), \mathfrak{g}_\mu(t)\big) = \big[\big(\mathfrak{e}_r \,.\, v_\mu(t),\ \mathfrak{e}_s \,.\, v_\mu(t)\big)\big]_{r,s} \cdot \big[\mathfrak{g}_{\mu,s}(t)\big]_s + \big[\big(\mathcal{L}_\mu\big(v_\mu(t)\big),\ \mathfrak{e}_r \,.\, v_\mu(t)\big)\big]_r = 0 \tag{6}$$

where $\mathfrak{g}_{\mu,s}$ denotes the sth component of $\mathfrak{g}_\mu$ with respect to the chosen basis. Depending on the specific problem, many different choices of space–time discretizations of the frozen system (5) are possible. To keep the notation simple, we will restrict ourselves to an explicit Euler time-stepping:

Definition 2.1. Let an H-dimensional space V_H and discrete parameter-dependent operators $\mathbb{L}_\mu$, $\mathbb{L}^G_{\mu,\mathfrak{h}}$ and $\mathbb{S}^G_r$ on V_H be given, approximating the operators $\mathcal{L}_\mu$, $\mathcal{L}_{\mu,\mathfrak{h}}$ and $\mathfrak{e}_r \,.\, (\cdot)$. Since $\mathrm{L}G$ acts on V by linear operators, we can assume that the $\mathbb{S}^G_i$ are linear. Furthermore let $P_H : V \longrightarrow V_H$ be a projection operator. For $K \in \mathbb{N}$, $\Delta t = T/K$ and $0 \leqslant k \leqslant K$, we define the discrete solutions $v^k_\mu \in V_H$, $\mathfrak{g}^k_\mu \in \mathrm{L}G$ of the frozen system (5) by the equations $v^0_\mu = P_H(u_0)$ and:

$$v^{k+1}_\mu = v^k_\mu - \Delta t\, \mathbb{L}^G_{\mu,\mathfrak{g}^k_\mu}\big(v^k_\mu\big), \qquad \big[\big(\mathbb{S}^G_r\big(v^k_\mu\big), \mathbb{S}^G_s\big(v^k_\mu\big)\big)\big]_{r,s} \cdot \big[\mathfrak{g}^k_{\mu,s}\big]_s = -\big[\big(\mathbb{L}_\mu\big(v^k_\mu\big), \mathbb{S}_r\big(v^k_\mu\big)\big)\big]_r \tag{7}$$

with $k = 0, \ldots, K - 1$. Moreover, let

$$g^0_\mu = 1_G, \qquad g^{k+1}_\mu = g^k_\mu \cdot \exp_G\big(\Delta t \cdot \mathfrak{g}^k_\mu\big), \quad k = 1, \ldots, n_t - 1.$$

An approximation of $u_\mu(t)$ is then given as $u^k_\mu = g^k_\mu \,.\, v^k_\mu$, with an appropriate discrete action of G.

3. The FrozenRB-scheme

After freezing and discretizing the original problem (1), we will now reduce the resulting system (7) by projecting it onto an N-dimensional reduced basis space $V_N \subseteq V_H$. To achieve fast online evaluation of the reduced scheme, we approximate the nonlinear operators $\mathbb{L}_\mu$, $\mathbb{L}^G_{\mu,\mathfrak{h}}$ using the method of *empirical operator interpolation* [2]. If $\hat{\varphi}_1, \ldots, \hat{\varphi}_H$ denotes a basis of the dual of V_H, this method produces indices $q_1, \ldots, q_M$ and vectors $\xi_1, \ldots, \xi_M, \xi^G_1, \ldots, \xi^G_M \in V_H$ such that:

$$\mathbb{L}_\mu(v) \approx \sum_{m=1}^{M} \hat{\varphi}_{q_m}\big(\mathbb{L}_\mu(v)\big) \cdot \xi_m, \qquad \mathbb{L}^G_{\mu,\mathfrak{h}}(v) \approx \sum_{m=1}^{M} \hat{\varphi}_{q_m}\big(\mathbb{L}^G_{\mu,\mathfrak{h}}(v)\big) \cdot \xi^G_m.$$

Given an appropriate projection operator $P_N : V_H \longrightarrow V_N$, the fully reduced scheme then reads:

$$\begin{cases} v^{k+1}_{N,\mu} = v^k_{N,\mu} - \Delta t\, P_N\left(\displaystyle\sum_{m=1}^{M} \hat{\varphi}_{q_m}\big(\mathbb{L}^G_{\mu,\mathfrak{g}^k_{N,\mu}}(v^k_{N,\mu})\big) \cdot \xi^G_m\right) \\ \big[\big(\mathbb{S}^G_r(v^k_{N,\mu}),\, \mathbb{S}^G_s(v^k_{N,\mu})\big)\big]_{r,s} \cdot \big[\mathfrak{g}^k_{N,\mu,s}\big]_s = -\left[\left(\displaystyle\sum_{m=1}^{M} \hat{\varphi}_{q_m}\big(\mathbb{L}_\mu(v^k_{N,\mu})\big) \cdot \xi_m,\, \mathbb{S}_r(v^k_{N,\mu})\right)\right]_r \end{cases} \tag{8}$$

with $v^k_{N,\mu} \in V_N$, $\mathfrak{g}^k_{N,\mu} \in \mathrm{LG}$, and $v^0_{N,\mu} = P_N(v^0_\mu)$. An appropriate reduced space V_N and the interpolation data q_m, ξ_m, and ξ^G_m can be constructed from solutions of (7) using reduced basis techniques developed in [2]. Choosing the same interpolation points for $\mathbb{L}_\mu$ and $\mathbb{L}^G_{\mu,\mathfrak{h}}$ ensures that expensive evaluations of nonlinear flux functions have to be carried out only once for both operators.

3.1. Offline/online decomposition

The discrete operators $\mathbb{L}_\mu$, $\mathbb{L}^G_{\mu,\mathfrak{h}}$ that arise from standard discretizations share the property of being local in the sense that for an appropriate basis φ_i of V_H each evaluation $\hat{\varphi}_i\ (\mathbb{L}_\mu(v))$ only depends on at most C degrees of freedom of v, independently of i, v and H (H-independent DOF dependence [2]). In particular, each of these evaluations can be performed at a speed independent of H. Under this assumption, we achieve fast online evaluation of (8) using precomputed data as follows:

Proposition 3.1. *Let $\mathbb{L}_\mu$, $\mathbb{L}^G_{\mu,\mathfrak{h}}$ have the H-independent DOF dependence property. Then there are indices $q'_1, \ldots, q'_L$ with $L \leqslant 2CM$ such that*:

$$\hat{\varphi}_{q_m}\big(\mathbb{L}_\mu(v)\big) = \hat{\varphi}_{q_m}\left(\mathbb{L}_\mu\left(\sum_{l=1}^{L} \hat{\varphi}_{q'_l}(v) \cdot \varphi_{q'_l}\right)\right), \qquad \hat{\varphi}_{q_m}\big(\mathbb{L}^G_{\mu,\mathfrak{h}}(v)\big) = \hat{\varphi}_{q_m}\left(\mathbb{L}^G_{\mu,\mathfrak{h}}\left(\sum_{l=1}^{L} \hat{\varphi}_{q'_l}(v) \cdot \varphi_{q'_l}\right)\right)$$

for all $v \in V_H$, $m = 1, \ldots, M$.

If $\psi_1, \ldots, \psi_N$ is a basis of V_N, define matrices $\mathbf{P}$, $\mathbf{EV}$, $\mathbf{PCL}^{r,s}$ and $\mathbf{PCR}^r$ for $1 \leqslant r, s \leqslant \dim G$ as:

$$\begin{aligned} &\mathbf{P}_{n,m} = \hat{\psi}_n\big(P_N(\xi^G_m)\big), \qquad &&\mathbf{PCL}^{r,s}_{n,n'} = \big(\mathbb{S}^G_r(\varphi_n),\, \mathbb{S}^G_s(\varphi_{n'})\big), \qquad &&1 \leqslant n, n' \leqslant N,\ 1 \leqslant m \leqslant M, \\ &\mathbf{EV}_{l,n} = \hat{\varphi}_{q'_l}(\psi_n), &&\mathbf{PCR}^r_{m,n} = \big(\xi_m,\, \mathbb{S}_r(\varphi_n)\big), &&1 \leqslant l \leqslant L. \end{aligned}$$

Let moreover $\mathbf{L}_\mu$, $\mathbf{L}^G_{\mu,\mathfrak{g}} : \mathbb{R}^L \longrightarrow \mathbb{R}^M$ be given as:

$$\big[\mathbf{L}_\mu(\mathbf{y})\big]_m = \hat{\varphi}_{q_m}\left(\mathbb{L}_\mu\left(\sum_{l=1}^{L} \mathbf{y}_l \cdot \varphi_{q'_l}\right)\right), \qquad \big[\mathbf{L}^G_{\mu,\mathfrak{g}}(\mathbf{y})\big]_m = \hat{\varphi}_{q_m}\left(\mathbb{L}^G_{\mu,\mathfrak{g}}\left(\sum_{l=1}^{L} \mathbf{y}_l \cdot \varphi_{q'_l}\right)\right).$$

If $\mathbf{v}^k_{N,\mu}$ is the coefficient vector of $v^k_{N,\mu}$ with respect to the reduced basis of V_N, then (8) is equivalent to:

$$\begin{cases} \mathbf{v}^{k+1}_{N,\mu} = \mathbf{v}^k_{N,\mu} - \Delta t\, \mathbf{P} \cdot \mathbf{L}^G_{\mu,\mathfrak{g}^k_{N,\mu}}\big(\mathbf{EV} \cdot \mathbf{v}^k_{N,\mu}\big) \\ \big[\big(\mathbf{v}^k_{N,\mu}\big)^T \cdot \mathbf{PCL}^{r,s} \cdot \mathbf{v}^k_{N,\mu}\big]_{r,s} \cdot \big[\mathfrak{g}^k_{N,\mu,s}\big]_s = -\big[\big(\mathbf{L}_\mu\big(\mathbf{EV} \cdot \mathbf{v}^k_{N,\mu}\big)\big)^T \cdot \mathbf{PCR}^r \cdot \mathbf{v}^k_{N,\mu}\big]_r\,. \end{cases}$$

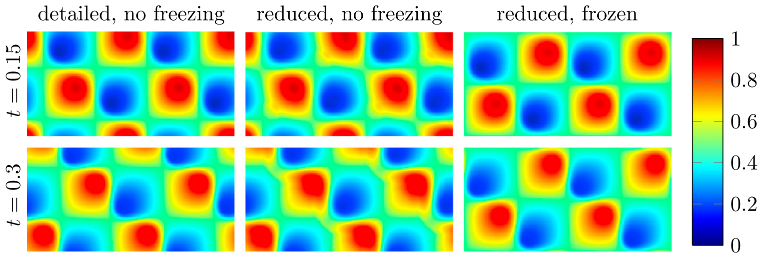

Fig. 1. Numerical solutions of the Burgers problem (9) for $\mu = 1.5$; left column: detailed approximation; center column: reduced approximation; right column: reduced approximation of the frozen solution v_μ (20 basis vectors, 38 interpolation points).

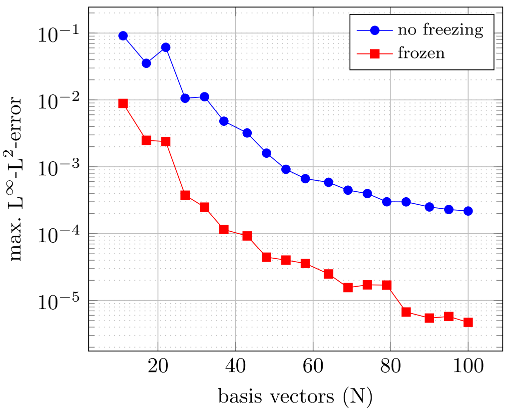

Fig. 2. Error of the FrozenRB-approximation of (10) in comparison with the same reduction method without freezing. The number of interpolation points M is given by $M = 1.8 \cdot N$. The error is estimated as the maximum approximation error over a set of 100 randomly chosen parameters.

4. Numerical experiment

As a first test for our new method, we consider the Burgers-type problem from [2,5], i.e. we solve:

$$\partial_t u_\mu(t) + \nabla \cdot \left(\mathbf{b}\, u_\mu(t)^\mu\right) = 0, \quad u_\mu(0) = u_0 \tag{9}$$

for $t \in [0, 0.3]$ on the domain $\Omega = [0,2] \times [0,1]$ identifying $\{0\} \times [0,1]$ with $\{2\} \times [0,1]$ and $[0,2] \times \{0\}$ with $[0,2] \times \{1\}$. The parameter μ is allowed to vary in the interval $[1,2]$. Moreover, let $\mathbf{b} = [1\ 1]^T$ and $u_0(x_1, x_2) = 1/2(1 + \sin(2\pi x_1)\sin(2\pi x_2))$. Eq. (9) is invariant under the action of the group $\mathbb{R}^2$ by translations of Ω. Since the action of $\mathrm{L}\mathbb{R}^2 = \mathbb{R}^2$ is given by negative spatial derivatives, this leads us to the frozen PDAE:

$$\begin{cases} \partial_t v_\mu(t) + \nabla \cdot \left(\mathbf{b}\, v_\mu(t)^\mu\right) - \mathfrak{g}_\mu(t) \cdot \nabla v_\mu(t) = 0 \\ \left[\left(\partial_{x_r} v_\mu(t),\ \partial_{x_s} v_\mu(t)\right)\right]_{r,s} \cdot \left[\mathfrak{g}_{\mu,s}(t)\right]_s = \left[\left(\nabla \cdot \left(\mathbf{b}\, v_\mu(t)^\mu\right),\ \partial_{x_r} v_\mu(t)\right)\right]_r, \quad 1 \leqslant r, s \leqslant 2 \end{cases} \tag{10}$$

with initial condition $v_\mu(0) = u_0$. The reconstruction equation for $g_\mu(t)$ is given by $\partial_t g_\mu(t) = \mathfrak{g}_\mu(t)$.

For the discretization of (10), we choose a monotone Lax–Friedrichs finite volume scheme on a 240×120 grid and take 200 steps in time. We use the EI–Greedy and POD–Greedy algorithms from [2] on a fixed training set of 10 equidistant parameters for the generation of the operator interpolation data and the reduced basis. In the EI–Greedy phase, a common interpolation basis $\xi_m = \xi_m^G$ of size 200 is created for the operators, followed by the creation of a reduced basis of size 100 using the POD–Greedy algorithm. For simplicity and to better evaluate the reduction potential of the FrozenRB-scheme, we directly compare the reduced simulations to the corresponding high-dimensional solution snapshots during basis generation. The integration of an error estimator should however be easily conceivable using the approaches of [2] and the references therein. Moreover, the scheme can be combined with the PODEI–Greedy algorithm [2] for simultaneous creation of the reduced basis and interpolation data. Fig. 1 shows that the FrozenRB-scheme provides already for 20 basis vectors a good approximation of v_μ (right column), whereas, for the same amount of basis vectors, the solutions of the same scheme without freezing are strongly deformed (center column). In Fig. 2 we compare the convergence of the schemes on the corresponding detailed approximations: the FrozenRB-scheme improves the approximation error for equal basis sizes by 1.7 orders of magnitude. Moreover, our experiments indicate an improved numerical stability of the frozen problem, which could be exploited to additionally reduce the number of time steps in the FrozenRB-scheme.

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