Differential geometry/Dynamical systems

A Semple-type approach to a problem of Goursat: The multi-flag case

Une approche de type Semple à un probléme de Goursat : le cas des multi-drapeaux

Alex L. Castro [a,1], Wyatt C. Howard [b,2]

[a] *Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro, Brazil*
[b] *Instituto National de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brazil*

A R T I C L E I N F O

A B S T R A C T

We consider the problem of classifying the orbits within a tower of fibrations with fibers diffeomorphic to projective planes and we generalize the tower of fiber bundles due to J. Semple. This tower, which was rediscovered by Montgomery and Zhitomirskii in the context of subriemannian geometry, admits a natural action of the diffeomorphism group of affine 3-space, and these orbits correspond to classes of Goursat multi-flags. We demonstrate that it is possible to classify many of these orbits by elementary means by appealing to some basic tools in projective geometry, and the combinatorics of spatial curves.

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R É S U M É

Nous nous intéressons au problème de classification des orbites dans une tour de fibration, où les fibres sont difféomorphes à des plans projectifs, généralisant ainsi les tours de fibrés projectif de J. Semple. Cette tour, redécouverte par R. Montgomery et M. Zhitomirskii dans le contexte de la géométrie sous-riemannienne, admet une action naturelle du groupe des difféomorphismes de l'espace affine de dimension 3, et ces orbites correspondent à des classes de multi-drapeaux de Goursat. Nous démontrons qu'il est possible de classifier un grand nombre de ces orbites de manière élémentaire en utilisant des outils classiques de géométrie projective et la combinatoire des courbes spatiales.

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1. Introduction

In this note, we generalize the Semple tower from enumerative algebraic geometry [9,7,6] to study the local theory of Goursat multi-flags. These are a class of nonholonomic distributions that generalizes the contact distributions of $J(n; k)$; the space of k-jets of maps from $\mathbb{R}$ to $\mathbb{R}^n$.

(A) Prolongation of a distribution. Prolongation d'une distribution différentielle ou champ de planes.

(B) Prolongation of a diffeomorphism. Prolongation d'un difféomorphisme.

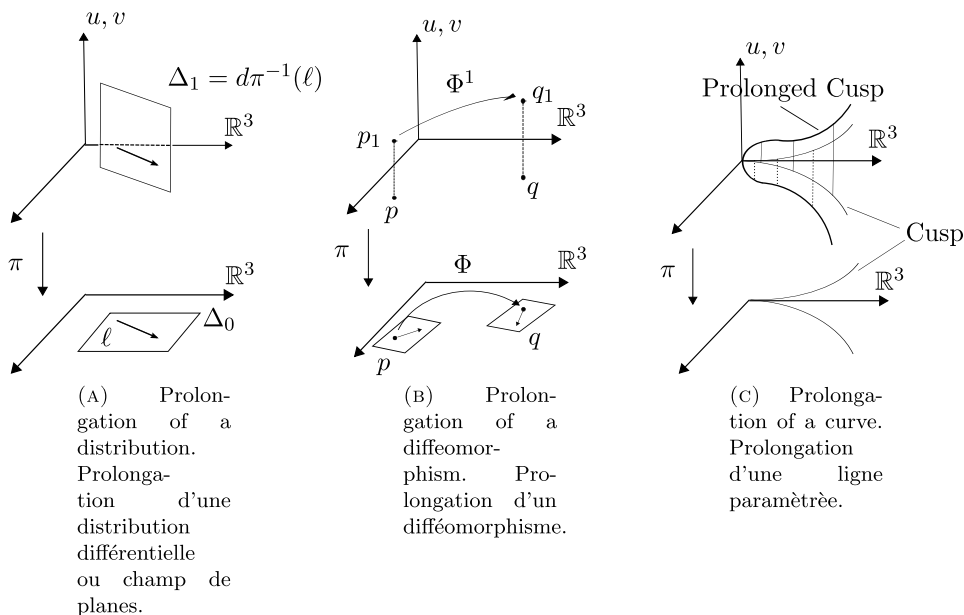(C) Prolongation of a curve. Prolongation d'une ligne paramètrèe.

Fig. 1. Prolongations.

Fig. 1. Prolongements.

In [4], we worked with a tower of fiber bundles, denoted by $S(n; k)$, containing the space $J(n; k)$ as an open dense subset. Each space $S(n; k)$ comes equipped with a geometric distribution Δ_k. The link between Goursat multi-flags and the Semple tower is the following:

the classification of Goursat multi-flags up to local equivalence is equivalent to the classification of points in the Semple tower $S(n; k)$ up to symmetry.

We present here the classification of Goursat 2-flags of small length, and investigate the orbits that arise from the Diff($\mathbb{A}^3$) action on $S(2; 4)$. The methods, however, are fully applicable to Goursat k-flags as well. For brevity, we shall restrict ourselves to a couple illustrative examples. Our main results are summarized in Theorem 3.1.

2. Definitions and statement of main results

A *geometric distribution* hereafter denotes a linear subbundle of the tangent bundle.

2.1. Prolongation

Let the pair (Z, Δ) denote a manifold Z of dimension d equipped with a distribution Δ of rank r. We denote by $\mathbb{P}(\Delta)$ the *projectivization* of Δ. As a manifold,

$$Z^1 := \mathbb{P}(\Delta)$$

has dimension $d + (r - 1)$.

Given an analytic horizontal curve $c : (I, 0) \to (Z, q)$, where I is some open interval in $\mathbb{R}$ containing the origin and $c(0) = q$, we can naturally define a new curve:

$$c^1 : (I, 0) \to (Z^1, (q, \ell))$$

with image in Z^1 and where $\ell = \text{span}\{\frac{dc}{dt}(0)\} \subset \Delta_q$. This new curve, $c^1(t)$, is called the *prolongation* of $c(t)$. This procedure can be iterated and the k-th prolongation of the curve $c(t)$ is denoted by $c^k(t)$.

The manifold Z^1 also comes equipped with a distribution Δ_1 called the *Cartan prolongation of Δ* [2]. Let $\pi : Z^1 \to Z$ be the projection map $(p, \ell) \mapsto p$. Then:

$$\Delta_1(p, \ell) = d\pi_{(p,\ell)}^{-1}(\ell).$$

It is easy to check that Δ_1 is also a distribution of rank r.

By a *symmetry* of the pair (Z, Δ) we mean a local diffeomorphism Φ of Z that preserves the subbundle Δ. The symmetries of (Z, Δ) can also be prolonged to symmetries Φ^1 of (Z^1, Δ_1) as follows. Define:

$$\Phi^1(p, \ell) := \big(\Phi(p), d\Phi_p(\ell)\big).$$

Since $d\Phi_p$ is invertible, the second component is well defined as a projective map. This new transformation of (Z^1, Δ_1) is called the *prolongation* of Φ. Prolongation of symmetries can also be iterated as well and the k-th prolongation is denoted by Φ^k. These prolongations are illustrated in Fig. 1.

2.2. Constructing the Semple Tower

Definition 2.1. The *Semple Tower* is a sequence of manifolds with distributions, $(S(n; k), \Delta_k)$, together with fibrations:

$$\cdots \to S(n; k) \to S(n; k - 1) \to \cdots \to S(n; 1) \to S(n; 0) = \mathbb{A}^{n+1}$$

and we write $\pi_{k,i} : S(n; k) \to S(n; i)$ for the respective bundle projections.

Definition 2.2. Diff(n) is taken to be the *pseudogroup of diffeomorphism germs of* $\mathbb{A}^n$.

The following result found in [10] by Shibuya and Yamaguchi will be important for our classification of points within the Semple Tower.

Theorem 2.1. *For $n > 1$ and $k > 0$ any local diffeomorphism of $S(n; k)$ preserving the distribution Δ_k is the restriction of the k-th prolongation of a local diffeomorphism $\Phi \in$ Diff(n).*

Shibuya and Yamaguchi also point out that this is a result due to A. Bäcklund in [3].

Remark 2.2. From now on we will write $S(k)$ to denote $S(2; k)$.

Definition 2.3. Two points p, q in $S(k)$ are said to be *equivalent*, written $p \sim q$, if there exists a $\Phi \in$ Diff(3) such that $\Phi^k(p) = q$.

Definition 2.4. We say that a curve or curve germ $c : (\mathbb{R}, 0) \to (\mathbb{A}^3, p_0)$ *realizes* the point $p_k \in S(k)$ if $c^k(0) = p_k$.

Definition 2.5. Two curves γ, σ in $\mathbb{R}^3$ are *RL equivalent*, written $\gamma \sim \sigma$, if there exists a diffeomorphism germ $\Phi \in$ Diff(3) and a reparametrization $\tau \in$ Diff$_0(1)$ such that $\sigma = \Phi \circ \gamma \circ \tau$.

3. Main results and ideas of proofs

From our recent work in [4] we have been able to completely classify the points within the first four levels of the tower. Our findings are summarized by the following.

Theorem 3.1 (*Orbit counting per level*). *In the $n = 2$ (or spatial) Semple Tower the number of orbits within each of the first four levels of the tower are as follows*:

– *Level* 1 *has* 1 *orbit,*
– *Level* 2 *has* 2 *orbits,*
– *Level* 3 *has* 7 *orbits,*
– *Level* 4 *has* 34 *orbits.*

3.1. Idea of proofs

The proofs use the singular curves approach first proposed in [8] as well as the **RC** coding system, which partitioned the various points within the tower [8,5]. One can then refine the *RC* coding system and define the *RVT* code, where the letter C is then replaced with either the letter V or the letters T or L with various decorations attached to them. If two points have different *RVT* codes, then they are necessarily inequivalent. We have highlighted the ways that these hyperplanes and distinguished lines arise; see Fig. 2. The concrete details about the *RVT* coding system can be found in [4]. We found that the curve approach did not suffice in classifying points within the fourth level and higher within the tower; this lead us to develop the *isotropy method*. The isotropy method helped us understand the incidence relations between the various critical hyperplanes and how these critical directions arise within the tower. Fig. 2a shows that there is only a single vertical
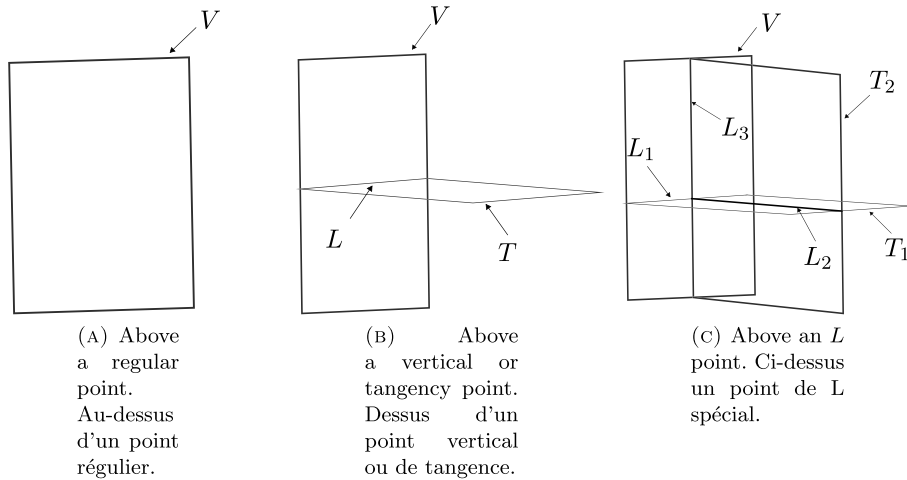
(A) Above a regular point. Au-dessus d'un point régulier.

(B) Above a vertical or tangency point. Dessus d'un point vertical ou de tangence.

(C) Above an L point. Ci-dessus un point de L spécial.

Fig. 2. Arrangement of critical hyperplanes.

Fig. 2. Arrangement d'hyperplans critiques.

hyperplane, the tangent space to the fiber, over a regular (R) point. Fig. 2b shows there are two critical hyperplanes over a vertical (V) or tangency (T) point. Fig. 2c tells us that over an L point there is a total of three critical hyperplanes. More details and specifics about how these hyperplanes arise within the tower can be found in [4] and [5]. While the isotropy method allows one to classify points within the various RVT classes within the Semple Tower, its only limitation is that it can be a computationally intensive process. As a result, we have begun programming and applying this method with Mathematica.

3.2. How the classification methods work in practice

3.2.1. The curve method illustrated: the class RVT

Any curve realizing a point in this class is either of the form $c_1(t) = (t^3, t^4, t^5)$ or $c_2(t) = (t^3, t^4, 0)$. To show this, we start with an analytic curve and looks at the restrictions on the coefficients in each of the component functions that come from the way the distribution Δ_3 is described in Kumpera Ruiz coordinates [5] and how the curves then project down to $\mathbb{R}^3$. Then we apply semigroup and basic singularity theory techniques outlined in [1] to see which terms can be eliminated from the projected curve in $\mathbb{R}^3$ to give the normal forms c_1 and c_2. Then since it can be shown that these two curves are not RL equivalent, and since prolongation and projection commute, this shows that when the two curves are prolonged to the third level of the tower and evaluated at $t = 0$, we will have two inequivalent points, i.e. $c_1^3(0) \nsim c_2^3(0)$. Hence the class RVT is comprised of two orbits.

3.2.2. The isotropy method illustrated: the class RVLT$_1$

We want to summarize the computations done for the class $RVLT_1$ using the notation from [4] for an RVT class that was not discussed in the paper. Using the isotropy method approach, our local diffeomorphisms are of the form $\Phi^3(x, y, z, u, v, u_2, v_2, u_3, v_3) = (\phi^1, \phi^2, \phi^3, \tilde{u}, \tilde{v}, \tilde{u}_2, \tilde{v}_2, \tilde{u}_3, \tilde{v}_3)$ with:

$$\tilde{u} = \frac{X^{(1)} \cdot \phi^2}{X^{(1)} \cdot \phi^1}, \qquad \tilde{u}_2 = \frac{X^{(2)} \cdot \phi^1}{X^{(2)} \cdot \tilde{u}}, \qquad \tilde{u}_3 = \frac{X^{(3)} \cdot \tilde{u}}{X^{(3)} \cdot \tilde{v}_2}$$

$$\tilde{v} = \frac{X^{(1)} \cdot \phi^3}{X^{(1)} \cdot \phi^1}, \qquad \tilde{v}_2 = \frac{X^{(2)} \cdot \tilde{v}}{X^{(2)} \cdot \tilde{u}}, \qquad \tilde{v}_3 = \frac{X^{(3)} \cdot \tilde{u}_2}{X^{(3)} \cdot \tilde{v}_2}.$$

Here, $X^{(1)} = u\frac{\partial}{\partial y} + v\frac{\partial}{\partial z} + \frac{\partial}{\partial x}$, $X^{(2)} = u_2 X^{(1)} + \frac{\partial}{\partial u} + v_2\frac{\partial}{\partial v}$, $X^{(3)} = u_3 X^{(2)} + v_3\frac{\partial}{\partial u_2} + \frac{\partial}{\partial v_2}$ and the restrictions on our component functions are $\phi_x^2(0) = 0$, $\phi_x^3(0) = 0$, and $\phi_y^3 = 0$. We apply the pushforward of Φ^3 to the line $\ell = b\frac{\partial}{\partial v_2} + c\frac{\partial}{\partial v_3}$, with $b, c \in \mathbb{R} - \{0\}^3$ in the distribution Δ_3 defined by:

$$dy - u\, dx = 0, \qquad dx - u_2\, du = 0, \qquad du - u_3\, dv_2 = 0$$

$$dz - v\, dx = 0, \qquad dv - v_2\, du = 0, \qquad du_2 - v_3\, dv_2 = 0,$$

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[3] This restriction on b and c is to ensure that the line ℓ is in the T_1 hyperplane.

and evaluating at $p_3 = (x, y, z, u, v, u_2, v_2, u_3, v_3)$ we get: $\Phi_*^3(\ell) = \text{span}\{b\frac{\phi_z^3(0)}{\phi_y^2(0)}\frac{\partial}{\partial v_2} + c\frac{(\phi_x^1(0))^2}{\phi_z^3(0)}\frac{\partial}{\partial v_3}\}$. Notice that $\phi_x^1(0) \neq 0$, $\phi_y^2(0) \neq 0$, and $\phi_z^3(0) \neq 0$. However, we can choose $\phi_x^1(0), \phi_y^2(0)$, and $\phi_z^3(0)$ to anything else other than zero. Since our distribution $\Delta_3(p_3)$ is coframed by dv_2, du_3, dv_3 and with $\ell' \equiv \Phi_*^3(\ell)$, we get $[du_2|_{\ell'} : du_3|_{\ell'} : dv_3|_{\ell'}] = [\frac{\phi_z^3(0)}{\phi_y^2(0)} : 0 : \frac{(\phi_x^1(0))^2}{\phi_z^3(0)}] = [1 : 0 : \frac{\phi_y^2(0)(\phi_x^1(0))^2}{(\phi_z^3(0))^2}]$. This gives a single orbit for the class $RVLT_1$ and we can represent this orbit by the direction $\ell = \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_3}$ in $\Delta_3(p_3)$.

4. Conclusion

Our work with the Semple Tower has been primarily focussed on classifying points up to symmetry at various levels within the tower. As was pointed out in [8], it is possible to end up with moduli appearing within various *RVT* classes. This means that there is a whole continuum's worth of points in a particular *RVT* class that are inequivalent. We are interested in determining at which level does this first occur in the $n = 2$ Semple Tower. We believe that one starting point is to look at the *RVT* classes of the form $RR^pVR^qVR^s$ for p, q, s nonnegative integers. During the summer of 2012 one of the authors worked on this problem with G. Kennedy (Ohio State) and S. Colley (Oberlin) and found that the class $RRVRRVRR\underline{R}$ has moduli occurring at the underlined R. Recently G. Kennedy and S. Colley contacted the authors concerning the appearance of moduli that were found in a number of experiment that they performed using the isotropy method. A subset of the computations showed that the following *RVT* classes have moduli occurring at the underlined R's: $RRVRRVRR\underline{R}$, $RRVRRRVRR\underline{R}$, and $RRRVRRRVRR\underline{R}RRR$. They have conjectured that when $p = 0$ there are no moduli, $p = 1$ there is a single modulus, $p = 2$ two moduli appear, and in general p moduli appear when $p > 1$. They formed this conjecture based on looking at the restrictions on the component functions within the isotropy group at each level in these *RVT* classes by using Mathematica. We have been working on this conjecture using the ideas of the isotropy method. More concretely, we are looking at the ways in which the local diffeomorphism group prolongs and acts on the points within these classes and the way these *RVT* classes place restrictions on the symmetries in the isotropy group. We believe that by understanding these examples it will help us better comprehend how moduli arise in other more complicated *RVT* codes.

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