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Global regularity and L^p-estimates for $\bar\partial$ on an annulus between two strictly pseudoconvex domains in a Stein manifold

Régularité globale et estimations L^p pour $\bar\partial$ sur une couronne entre deux domaines strictement pseudo-convexes dans une variété de Stein

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ABSTRACT

In this note, we prove an L^2-existence theorem for the $\bar\partial$-Neumann operator and the regularity for the $\bar\partial$-equation on an annulus type domain $D = D_1 \setminus \overline{D_2}$, where D_1 and D_2 are strictly pseudoconvex domains with smooth boundaries in a Stein manifold X of complex dimension $n \geqslant 3$, such that $\overline{D_2} \subset D_1 \Subset X$. Moreover, we obtain Hölder and L^p estimates for the $\bar\partial$-equation on strictly pseudoconcave domains with smooth $\mathcal{C}^3$-boundaries in X.

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RÉSUMÉ

Dans cette Note, nous démontrons un théorème d'existence L^2 pour l'opérateur de Neumann $\bar\partial$ et la régularité globale au bord de l'équation $\bar\partial$ sur de domaine de type couronne $D = D_1 \setminus \overline{D}_2$ où D_1 et D_2 sont des domaines strictement pseudo-convexes dont les bords sont régulés dans une variété de Stein X de dimension complexe $n \geqslant 3$, tels que $\overline{D}_2 \subset D_1 \Subset X$. De plus, nous obtenons des estimations de Hölder et L^p, $1 \leqslant p \leqslant \infty$, pour $\bar\partial$ sur des domaines strictement pseudo-concaves de frontière $\mathcal{C}^3$ dans X.

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1. Introduction and main results

The study of the $\bar\partial$-Neumann problem and the regularity of the $\bar\partial$-equation have attracted a lot of attention (see e.g., [10,15,11], and [7]). More precisely, by introducing the weighted $\bar\partial$-Neumann operator, Kohn [11] solved extensively this problem on bounded pseudoconvex domain in a complex manifold with smooth boundary, which can be exhausted by strictly pseudoconvex domains.

On the annulus type domain $M = M_1 \setminus \overline{M_2}$ between two smooth pseudoconvex domains M_1 and M_2 in $\mathbb{C}^n$ such that $\overline{M_2} \subset M_1 \Subset \mathbb{C}^n$, the closed range property and the global boundary regularity for $\bar\partial$ were studied by Shaw [15] for

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(r,s)-forms, where $0 \leqslant r \leqslant n$, $1 \leqslant s, \leqslant n-2$ and $n \geqslant 3$. The critical case when $s = n-1$ was established in Shaw [16]. In this case, the space of harmonic forms is infinite dimensional.

Motivated by the same problem for an annulus-type domain between two smooth strictly q-convex domains in $\mathbb{C}^n$, Khidr and Abdelkader [10] showed that the $\bar\partial$-operator has closed range in the L^2-setting and then they proved the L^2-existence theorem for the $\bar\partial$-Neumann operator for $\bar\partial$-closed (r,s)-forms, where $r \geqslant 0$ and $q \leqslant s < n-q-1$.

As a result, the exact regularity for the Bergman projection and the $\bar\partial$-Neumann operators is proved. Consequently, they proved that for a $\bar\partial$-closed (r,s)-form ($r \geqslant 0$ and $q \leqslant s < n-q-1$) with $\mathcal{C}^\infty$ coefficients smooth up to the boundary, there exists a $(r,s-1)$-form, smooth up to the boundary, which solves the $\bar\partial$-equation on such annuli domains.

In addition, estimates in Hölder and L^p-categories, $1 \leqslant p \leqslant \infty$, for solutions of the $\bar\partial$-equation are obtained for $\bar\partial$-closed (r,s)-form, $q \leqslant s < n-q-1$, on strictly q-concave domains with smooth $\mathcal{C}^3$ boundaries in $\mathbb{C}^n$.

It is now essential to extend the results in [10] to annuli-type domains in a Stein manifold and vector bundle-valued forms, which is the main aim of the present note. The solvability and regularity of the $\bar\partial$-equation on domains of complex manifolds have been discussed by several authors (see e.g., [5,7,6,4,9,1,2,13,18]).

We first fix the following standard notation. Let X be a Stein manifold of complex dimension $n \geqslant 3$ and $E \xrightarrow{\pi} X$ be a holomorphic Hermitian vector bundle of rank N over X, whose dual is $E^\star$. Let $\{U_j\}$; $j \in I$, be an open covering of X consisting of coordinates neighborhoods U_j with holomorphic coordinates $z_j = (z_j^1, z_j^2, \ldots, z_j^n)$ over which E is trivial, namely $\pi^{-1}(U_j) = U_j \times \mathbb{C}^N$. Let Ω be a relatively compact domain in X with smooth boundary $\partial\Omega$. Denote by $\Lambda^{r,s}(\Omega, E)$, $0 \leqslant r \leqslant n$; $0 \leqslant s \leqslant n$, the space of E-valued forms of type (r,s) and of class C^∞ on Ω and $L^2_{r,s}(\Omega, E)$ the Hilbert space obtained by completing the space $\Lambda_c^{r,s}(\Omega, E)$ of forms in $\Lambda^{r,s}(\Omega, E)$ with compact supports in Ω under the norm associated with the scalar product defined by the Hermitian metrics on X and on the fibers of E. $\Lambda^{r,s}(\overline{\Omega}, E)$ denotes the subspace of $\Lambda^{r,s}(\Omega, E)$ consisting of those (r,s)-forms that can be extended smoothly up to and including the boundary. Let $\bar\partial : L^2_{r,s}(\Omega, E) \to L^2_{r,s+1}(\Omega, E)$ be the weak maximal closed extension of the original $\bar\partial$ on C^∞-forms and $\bar\partial^* : L^2_{r,s+1}(\Omega, E) \to L^2_{r,s}(\Omega, E)$ be its L^2-adjoint. Let $\Box_{r,s} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : \mathrm{Dom}(\Box_{r,s}) \to L^2_{r,s}(\Omega, E)$ be the corresponding complex Laplacian and $\mathcal{H}_{r,s}(\Omega, E) = \{\alpha \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*) \subset L^2_{r,s}(\Omega, E) \mid \bar\partial\alpha = \bar\partial^*\alpha = 0\}$ be the space of harmonic forms. The Banach spaces $L^p_{r,s}(\Omega, E)$; $1 \leqslant p \leqslant \infty$, the Sobolev spaces $W^k_{r,s}(\Omega, E)$; $k \geqslant 0$, and the norms on these spaces are defined in the usual manner.

Finally, for $1 \leqslant p < \infty$ and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, we denote by $L^{p'}_{n-r,n-s}(\Omega, E^\star)$ the dual space to $L^p_{r,s}(\Omega, E)$. Our main results are as follows:

Theorem 1.1. *Let X be a Stein manifold of complex dimension n and $E \xrightarrow{\pi} X$ be a holomorphic Hermitian vector bundle of rank N over X. Let $D = D_1 \setminus \overline{D_2}$ be an annulus domain between two strictly pseudoconvex domains D_1 and D_2 with smooth boundaries such that $\overline{D_2} \subset D_1 \Subset X$. Then for every $\bar\partial$-closed form f in $\Lambda^{r,s}(\overline{D}, E)$, $f \perp \mathcal{H}_{r,s}(D, E)$, where $0 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n-2$ and $n \geqslant 3$, there exists a form u in $\Lambda^{r,s-1}(\overline{D}, E)$ such that $\bar\partial u = f$.*

For $s = n-1$, we assume that the restriction of f to ∂D_2 satisfies the compatibility condition:

$$\int_{\partial D_2} f \wedge \phi = 0 \quad \text{for every } \phi \in \Lambda^{n-r,0}(D_2, E^\star) \cap \mathrm{Ker}(\bar\partial)$$

and the same conclusion holds.

Theorem 1.2. *Let $D = D_1 \setminus \overline{D}_2 \Subset X$ be the annulus domain between two strictly pseudoconvex domains D_1 and D_2 with smooth $\mathcal{C}^3$ boundaries in an n-dimensional Stein manifold X such that $\overline{D_2} \subset D_1 \Subset X$ and $E \xrightarrow{\pi} X$ be a holomorphic Hermitian vector bundle of rank N over X. Then for every $\bar\partial$-closed form f in $L^1_{r,s}(D, E)$, $0 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n-2$ and $n \geqslant 3$, there exists a form g in $L^1_{r,s-1}(D, E)$ such that $\bar\partial g = f$. If f is $\mathcal{C}^\infty$, then g is also $\mathcal{C}^\infty$. Moreover, if f is in $L^p_{r,s}(D, E)$, $1 \leqslant p \leqslant \infty$, then g is in $L^p_{r,s-1}(D, E)$ and satisfies the estimates:*

$$\|g\|_{L^p_{r,s-1}(D,E)} \leqslant K \|f\|_{L^p_{r,s}(D,E)}; \quad 1 \leqslant p \leqslant \infty. \tag{1.1}$$

When $s = n-1$, if we assume that the restriction of f to ∂D_2 satisfies:

$$\int_{\partial D_2} f \wedge \phi = 0 \quad \text{for every } \phi \in L^{p'}_{n-r,0}(D_2, E^\star) \cap \mathrm{Ker}(\bar\partial),$$

then the same statement and (1.1) hold for all $1 \leqslant p < \infty$.

In addition, if $f \in L^\infty_{r,s}(D, E)$, $0 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n-2$, then g satisfies the $\frac{1}{2}$-Hölder estimate

$$\|g(z) - g(z')\| \leqslant K |z - z'|^{\frac{1}{2}} \|f\|_{L^\infty_{r,s}(D,E)}, \quad (z, z') \in D \times D,$$

the solution g depends linearly on f and the positive constant $K > 0$ is independent of f, p, and small $\mathcal{C}^3$ perturbations of D.

2. Proof of Theorem 1.1

The proof is given in several steps throughout the rest of this section. We first extend the $\frac{1}{2}$-subelliptic estimate proved in [10] to E-valued forms.

Let $\omega_1, \dots, \omega_N$ be an orthonormal basis on $E_z = \pi^{-1}(z)$ for every, $z \in U_j$; $j \in I$. Thus every E-valued (r,s)-form f can be written locally, on U_j, as $f(z) = \sum_{\mu=1}^{N} f^\mu(z)\omega_\mu(z)$, where f^μ are the components of the restriction of f on U_j. Since ∂D is compact, there exists a finite number of elements of the covering $\{U_j\}$, say, U_{j_ν}; $\nu = 1, 2, \dots, m$ such that $\bigcup_{\nu=1}^{m} U_{j_\nu}$ cover ∂D. Let U be a small neighborhood of a given boundary point $\zeta \in \partial D_1$ (or $\zeta \in \partial D_2$) such that $U \Subset V \Subset U_{j_\nu}$, for a certain $j_\nu \in I$. If $f \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*) \cap L^2_{r,s}(D,E)$, $0 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n-2$, on applying the $\frac{1}{2}$-subelliptic estimate of [10] to each f^μ and adding for $\mu = 1, \dots, N$, we get $\frac{1}{2}$-subelliptic estimate for $f|_{D\cap U_{j_\nu}}$. Using a partition of unity, we obtain the estimate:

$$\|f\|_{W^{\frac{1}{2}}_{r,s}(D,E)} \leqslant C\big(\|\bar\partial f\|^2_{L^2_{r,s+1}(D,E)} + \|\bar\partial^* f\|^2_{L^2_{r,s-1}(D,E)} + \|f\|^2_{L^2_{r,s}(D,E)}\big). \tag{2.1}$$

The estimate (2.1) and arguments similar to those in [5] and [10] imply the following L^2-existence theorem for the $\bar\partial$-Neumann operator.

Theorem 2.1. *Let the situations be as in Theorem 1.1. Then for each $0 \leqslant r \leqslant n$ and $1 \leqslant s \leqslant n-2$, the $\bar\partial$-Neumann operator $N_{r,s} : L^2_{r,s}(D,E) \to L^2_{r,s}(D,E)$ exists and $\mathcal{H}_{r,s}(D,E)$ is of finite dimension. Moreover, for every $f \in L^2_{r,s}(D,E)$, we have the orthogonal decomposition*:

$$f = \bar\partial\bar\partial^\star N_{r,s} f + \bar\partial^\star\bar\partial N_{r,s} f + H_{r,s} f \tag{2.2}$$

where $H_{r,s} : L^2_{r,s}(D,E) \to \mathcal{H}_{r,s}(D,E)$ is the orthogonal projection.

In addition, the $\bar\partial$-Neumann operator $N_{r,s}$ satisfies the following properties:

(1) *$N_{r,s}H_{r,s} = H_{r,s}N_{r,s} = 0$. $N_{r,s}\Box_{r,s} = \Box_{r,s}N_{r,s} = Id - H_{r,s}$ on $\mathrm{Dom}(\Box_{r,s})$, where Id is the identity operator.*
(2) *$\bar\partial N_{r,s-1} = N_{r,s}\bar\partial$ on $\mathrm{Dom}(\bar\partial)$ and $\bar\partial^\star N_{r,s+1} = N_{r,s}\bar\partial^\star$ on $\mathrm{Dom}(\bar\partial^\star)$.*
(3) *If (2.2) holds, then if $\bar\partial f = 0$ and $f \perp \mathcal{H}_{r,s}(D,E)$, so $f = \bar\partial\bar\partial^\star N_{r,s} f$ and $u = \bar\partial^\star N_{r,s} f$ is the unique solution to the equation $\bar\partial u = f$ which is orthogonal to $\mathrm{Ker}(\bar\partial)$ with $\|u\|_{L^2_{r,s-1}(D,E)} \leqslant C_s\|f\|_{L^2_{r,s}(D,E)}$.*
(4) *$N_{r,s}(\Lambda^{r,s}(\overline{D},E)) \subseteq \Lambda^{r,s}(\overline{D},E)$, and for each $k \in \mathbb{R}$ there is a positive constant C_s such that the estimate*

$$\|N_{r,s} f\|_{W^{k+1}_{r,s}(D,E)} \leqslant C_s\|f\|_{W^k_{r,s}(D,E)}$$

holds for all $f \in \Lambda^{r,s}(\overline{D},E)$.

Let D be given as in Theorem 1.1. By the same way for bounded pseudoconvex domains, we recall that a differential operator is said to be globally regular if it maps the space $\Lambda^{r,s}(\overline{D},E)$ continuously to itself, and is called exactly regular if it maps the Sobolev spaces $W^k_{r,s}(D,E)$, $k \geqslant 0$, to themselves. The Bergman projections $P_{r,s}$ are the orthogonal projections of the space of square integrable (r,s)-forms onto the subspace of $\bar\partial$-closed (r,s)-forms. A direct relation between the Bergman projection and the $\bar\partial$-Neumann operator $N_{r,s}$ is given by Kohn's formulas $P_{r,s-1} = Id - \bar\partial^* N_{r,s}\bar\partial$ for $r \geqslant 0$ and $1 \leqslant s \leqslant n-2$. We refer to [17] for more information on the relationship between the regularity properties of the $\bar\partial$-Neumann operators and the Bergman projections.

Theorem 2.2. *Let X, E, and D be given as in Theorem 1.1, then the Bergman projection $P_{r,s-1}$ and the $\bar\partial$-Neumann operator $N_{r,s}$ are continuous on the Sobolev spaces $W^k_{r,s-1}(D,E)$ and $W^k_{r,s}(D,E)$ respectively for all $0 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n-2$, and $k \geqslant 0$. Moreover, there exists a constant $C_k > 0$ such that*:

$$\|N_{r,s} f\|_{W^k_{r,s}(D,E)} \leqslant C_k\|f\|_{W^k_{r,s}(D,E)}; \quad f \in W^k_{r,s}(D,E)$$
$$\|P_{r,s-1} f\|_{W^k_{r,s-1}(D,E)} \leqslant C_k\|f\|_{W^k_{r,s-1}(D,E)}; \quad f \in W^k_{r,s-1}(D,E)$$

Proof. The proof follows by applying Theorem 2.2 in [10] to the components f^μ of f as in the proof of the estimate (2.1), and using the density of $\Lambda^{r,s}(\overline{D},E)$ in $W^k_{r,s}(D,E)$ (see [14, Theorem 3.29]). □

Corollary 2.3. *Let X, E, and D be given as above, then the canonical solution operators $\bar\partial N_{r,s}$, $\bar\partial^* N_{r,s}$, and $\bar\partial^*\bar\partial N_{r,s}$ (and hence $\bar\partial\bar\partial^* N_{r,s} = P_{r,s}$), $0 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n-2$, are exactly regular as the $\bar\partial$-Neumann operator $N_{r,s}$ is.*

Theorem 2.4. *Under the above assumptions, the $\bar\partial$-Neumann operator $N_{r,s}$ is exactly regular for all $0 \leqslant r \leqslant n$ and $1 \leqslant s \leqslant n-2$ if and only if the Bergman projections $P_{r,s-1}$, $P_{r,s}$ and $P_{r,s+1}$ are exactly regular.*

Proof. The proof follows from the idea in [3] and depends on the Kohn's weighted L^2-theory for the $\bar\partial$-Neumann operator in [11]. $\square$

By Theorem 2.1(3), Theorem 2.2, and the density of $\Lambda^{r,s}(\overline{D},E)$ in $W^k_{r,s}(D,E)$, the following corollary is immediate.

Corollary 2.5. *Let X, E, and D be given as above, then for every $\bar\partial$-closed form f in $W^k_{r,s}(D,E)$, $f\perp\mathcal{H}_{r,s}(D,E)$, where $0\leqslant r\leqslant n$, $1\leqslant s\leqslant n-2$, and $k\geqslant 0$, there exists a form u in $W^k_{r,s-1}(D,E)$ that solves the equation $\bar\partial u=f$ and satisfies the estimates $\|u\|_{W^k_{r,s-1}(D)}\leqslant C_k\|f\|_{W^k_{r,s}(D)}$.*

When $s=n-1$, if we assume furthermore that the restriction of f to ∂D_2 satisfies the condition $\int_{\partial D_2} f\wedge\phi=0$ for every $\phi\in L^2_{n-r,0}(D_2,E^\star)\cap\operatorname{Ker}(\bar\partial)$, then the same conclusion holds.

End proof of Theorem 1.1. By Corollary 2.5, for each $k\geqslant 0$, there exists $u_k\in W^k_{r,s-1}(D,E)$ such that $\bar\partial u_k=f$. We modify each u_i by an element of $\operatorname{Ker}(\bar\partial)$ in order to construct a telescoping series that belongs to $W^k_{r,s}(D,E)$ for each $k\geqslant 1$. To conclude the proof, we first claim that $W^k_{r,s}(M)\cap\operatorname{Ker}(\bar\partial)$ is dense in $W^m_{r,s}(D,E)\cap\operatorname{Ker}(\bar\partial)$ for any $k>m\geqslant 0$. Since $\Lambda^{r,s}(\overline{D},E)$ is dense in $W^m_{r,s}(D,E)$, $m\geqslant 0$, in the $W^m_{r,s}(D,E)$-norm, then for a given $\eta\in W^m_{r,s}(D,E)\cap\operatorname{Ker}(\bar\partial)$, there is a sequence $\eta_j\in\Lambda^{r,s}(\overline{D},E)$ that converges to η in the $W^m_{r,s}(D,E)$-norm, i.e. $\|\eta_j-\eta\|_{W^m_{r,s}(D,E)}\to 0$ as $j\to\infty$. $\bar\partial\eta=0$ implies that $\eta-P_{r,s}\eta=\bar\partial^* N_{r,s+1}\bar\partial\eta=0$, so $\eta=P_{r,s}\eta$. Let $\hat\eta_j=P_{r,s}\eta_j$. $\hat\eta_j\in W^k_{r,s}(D,E)\cap\operatorname{Ker}(\bar\partial)$, since the Bergman projection $P_{r,s}$ is a bounded operator on $W^k_{r,s}(D,E)$. By the same reason, we have $\|\hat\eta_j-\eta\|_{W^m_{r,s}(D,E)}=\|P_{r,s}(\eta_j-\eta)\|_{W^m_{r,s}(D,E)}\leqslant C\|\eta_j-\eta\|_{W^m_{r,s}(D,E)}\to 0$ as $j\to\infty$. This implies that $\hat\eta_j\to\eta$ in the $W^m_{r,s}(D,E)$-norm. Thus, indeed, $W^k_{r,s}(D,E)\cap\operatorname{Ker}(\bar\partial)$ is dense in $W^m_{r,s}(D,E)\cap\operatorname{Ker}(\bar\partial)$ for any $k>m\geqslant 0$.

Next, using this result and on following the inductive arguments due to [12, p. 230], we construct a sequence $\widetilde{u}_k\in W^k_{r,s-1}(D,E)$, $\bar\partial\widetilde{u}_k=f$, and $\|\widetilde{u}_{k+1}-u_k\|_{W^k_{r,s-1}(D,E)}\leqslant 2^{-k}$ as follows:

$$\widetilde{u}_1=u_1,\qquad \widetilde{u}_2=u_2+v_2,$$

where $v_2\in W^2_{r,s-1}(D,E)\cap\operatorname{Ker}(\bar\partial)$ is such that:

$$\|\widetilde{u}_2-u_1\|_{W^1_{r,s-1}(D,E)}\leqslant 2^{-1}$$

and in general:

$$\widetilde{u}_{i+1}=u_{i+1}+v_{i+1}$$

where $v_{k+1}\in W^{k+1}_{r,s-1}(D,E)\cap\operatorname{Ker}(\bar\partial)$ is such that:

$$\|\widetilde{u}_{k+1}-u_k\|_{W^k_{r,s-1}(D,E)}\leqslant 2^{-k}.$$

Clearly $\bar\partial\widetilde{u}_k=f$, so set:

$$u=\widetilde{u}_j+\sum_{k=j}^{\infty}(\widetilde{u}_{k+1}-\widetilde{u}_k),\qquad j\in\mathbb{N}.$$

It follows that $u\in W^k_{r,s-1}(D,E)$ for each $k\in\mathbb{N}$, and hence $u\in\Lambda^{r,s-1}(D,E)$ and $\bar\partial u=f$. The general case is obtained from an interpolation of linear operators. By the Sobolev embedding theorem, $u\in\Lambda^{r,s-1}(\overline{D},E)$. The proof is complete. $\square$

3. Proof of Theorem 1.2

The proof is also given in several steps throughout the rest of this section. We begin by extending the local result in [10] to the current case. Let $D=D_1\setminus\overline{D}_2\Subset X$ be the annulus domain between two strictly pseudoconvex domains D_1 and D_2 in an n-dimensional Stein manifold X and $\rho:\mathbb{U}\to\mathbb{R}$ be a $\mathcal{C}^3$ defining function of D_2. Let $\zeta_0\in\partial D_2$ be an arbitrary fixed point. Then there exists $U_j\Subset V\Subset\mathbb{U}$, for a certain $j\in I$, such that $\zeta_0\in U_j$. Let $z_j:U_j\to\mathbb{C}^n$ be a holomorphic coordinate on U_j such that $z_j(\zeta_0)=0$; there is a neighborhood $W_\delta\subset U_j$ of ζ_0 such that $z_j(W_\delta)=B(0,\delta)$ is the open ball of center 0 and radius δ in $\mathbb{C}^n$. Thus, by Theorem 3.2 in [10], there is a local integral solution operator T^s solves the $\bar\partial$-equation for $\bar\partial$-closed (r,s)-forms f with L^p-coefficients on strictly pseudoconcave domain $G\Subset\mathbb{C}^n$. The resulting solution operator T^s is then pulled back locally to an open set D_{ζ_0} with $W_{\delta/2}\cap D\subset D_{\zeta_0}\subset W_\delta\cap D$. If $f\in L^p_{r,s}(D,E)$, $0\leqslant r\leqslant n$, $1\leqslant s\leqslant n-2$, on applying Theorem 3.2 of [10] to each f^μ, $\mu=1,\dots,N$, and writing $T^s f=\sum_{\mu=1}^N (T^s f)^\mu\omega_\mu$, we then conclude the following local result.

Theorem 3.1. *Under the above assumptions, for every $\bar\partial$-closed form $f \in L^1_{r,s}(D_{\zeta_0}, E)$, $0 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n-2$, there exists a form $u = T^s f \in L^1_{r,s-1}(D_{\zeta_0}, E)$ such that $\bar\partial u = f$. If f is $\mathcal{C}^\infty$, then u is $\mathcal{C}^\infty$. Moreover, if $f \in L^p_{r,s}(D_{\zeta_0}, E)$, $1 \leqslant p \leqslant \infty$, then there exists a constant $K = K(N, s) > 0$ such that:*

$$\|u\|_{L^p_{r,s-1}(D_{\zeta_0},E)} \leqslant K\|f\|_{L^p_{r,s}(D_{\zeta_0},E)}. \tag{3.1}$$

When $s = n-1$, if we assume that the restriction of f to ∂D_{ζ_0} satisfies:

$$\int\limits_{\partial D_{\zeta_0}} f \wedge \phi = 0 \quad \text{for every } \phi \in L^{p'}_{n-r,0}\big(D_{\zeta_0}, E^\star\big) \cap \operatorname{Ker}(\bar\partial),$$

then the same statement and (3.1) hold for all $1 \leqslant p < \infty$.

Furthermore, if $f \in L^\infty_{r,s}(D_{\zeta_0}, E)$, $0 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n-2$, then u satisfies the $\frac{1}{2}$-Hölder estimate

$$\big\|u(z) - u\big(z'\big)\big\| \leqslant K\big|z - z'\big|^{\frac{1}{2}}\|f\|_{L^\infty_{r,s}(D_{\zeta_0},E)}, \quad \big(z, z'\big) \in D_{\zeta_0} \times D_{\zeta_0}.$$

Theorem 3.1 yields the following extension lemma which enables us to complete the proof of Theorem 1.2.

Lemma 3.2. *Let X, E, and D be given as in Theorem 1.2, then there exists another slightly larger C^3 strictly pseudoconcave domain $\widehat{D} \Subset X$ such that $\bar D \Subset \widehat{D}$ and for every $\bar\partial$-closed $f \in L^1_{r,s}(D, E)$, $0 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n-2$, there exist two bounded linear operators L_1, L_2, a form $\widehat{f} = L_1 f \in L^1_{r,s}(\widehat{D}, E)$ and a form $\chi = L_2 f \in L^1_{r,s-1}(D, E)$ such that:*

(i) $\bar\partial\widehat{f} = 0$ *in* $\widehat{D}$.
(ii) $\widehat{f} = f - \bar\partial\chi$ *in* D.
(iii) *If f is in $L^p_{r,s}(D, E)$, then $\widehat{f}$ is in $L^p_{r,s}(\widehat{D}, E)$ and χ is in $L^p_{r,s-1}(D, E)$ with the estimates:*

$$\|\widehat{f}\|_{L^p_{r,s}(\hat D,E)} \leqslant K\|f\|_{L^p_{r,s}(D,E)}, \quad 1 \leqslant p \leqslant \infty,$$
$$\|\chi\|_{L^p_{r,s-1}(D,E)} \leqslant K\|f\|_{L^p_{r,s}(D,E)}, \quad 1 \leqslant p \leqslant \infty.$$

For $s = n-1$, if we assume that the restriction of f to ∂D_2 satisfies:

$$\int\limits_{\partial D_2} f \wedge \phi = 0 \quad \text{for every } \phi \in L^{p'}_{n-r,0}\big(D_2, E^\star\big) \cap \operatorname{Ker}(\bar\partial),$$

then the same statements (i), (ii), *and* (iii) *hold for all $1 \leqslant p < \infty$.*
If $f \in L^\infty_{r,s}(D, E)$, $0 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n-2$, then χ satisfies the $\frac{1}{2}$-Hölder estimate:

$$\big\|\chi(z) - \chi\big(z'\big)\big\| \leqslant K\big|z - z'\big|^{\frac{1}{2}}\|f\|_{L^\infty_{r,s}(D,E)}, \quad \big(z, z'\big) \in D \times D,$$

the constant K is independent of f, p and small C^3 perturbation of D. If f is $\mathcal{C}^\infty$, then $\widehat{f}$ and χ are also $\mathcal{C}^\infty$.

End proof of Theorem 1.2. Let $\widehat{D}$, $\widehat{f}$ and χ be as in Lemma 3.2. According to Theorem 6.6 of Henkin and Leiterer [6], there exists a strictly pseudoconcave domain D' with smooth boundary such that:

$$D \Subset D' \Subset \widehat{D}.$$

The solvability with L^2-estimates for $\bar\partial\chi = \widehat{f}$ on D' follows from the following L^2-existence theorem for $\bar\partial$ which follows as in Hörmander [7] (see also [1]).

Theorem 3.3. *Let X, D, and E be given as above. Then, for every $\bar\partial$-closed form f in $L^2_{r,s}(D, E)$, $0 \leqslant r \leqslant n$, $q \leqslant s \leqslant n-2$, there exist a form χ in $L^2_{r,s-1}(D, E)$ and a constant $C > 0$ such that $\bar\partial\chi = f$ in D and $\|\chi\|_{L^2_{r,s-1}(D,E)} \leqslant C\|f\|_{L^2_{r,s}(D,E)}$.*

When $s = n-1$, if f is in $L^2_{r,n-1}(D, E)$ and its restriction to ∂D_2 satisfies $\int_{\partial D_2} f \wedge \phi = 0$ for every ϕ in $L^2_{n-r,0}(D_2, E^\star) \cap \operatorname{Ker}(\bar\partial)$, then there exist a form χ in $L^2_{r,n-2}(D, E)$ and a constant $C > 0$ such that $\bar\partial\chi = f$ in D and $\|\chi\|_{L^2_{r,n-2}(D,E)} \leqslant C\|f\|_{L^2_{r,n-1}(D,E)}$.

Finally, the interior regularity properties for solutions of the $\bar\partial$-equation and Lemma 3.2 imply Theorem 1.2 with L^p-estimates for the intermediate cases $2 \leqslant p \leqslant \infty$. The estimates for $1 \leqslant p < 2$ follow from the proof of Theorem 1.2.2 in [8]. This completes the proof of Theorem 1.2. □

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