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Numerical analysis

Approximation by Müntz spaces on positive intervals

Approximation par espaces de Müntz sur un intervalle positif

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ABSTRACT

The so-called Bernstein operators were introduced by S.N. Bernstein in 1912 to give a constructive proof of Weierstrass' theorem. We show how to extend his result to Müntz spaces on positive intervals.

RÉSUMÉ

En 1912, les opérateurs dits de Bernstein permirent à S.N. Bernstein de donner une preuve constructive du théorème de Weierstrass. Nous étendons ce résultat aux espaces de Müntz sur des intervalles positifs.

1. Introduction

The famous Bernstein operator $\mathbb{B}_k$ of degree k on a given non-trivial interval $[a,b]$, associates with any $F \in C^0([a,b])$ the polynomial function:

$$\mathbb{B}_k F(x) := \sum_{i=0}^{k} F\left(\left(1-\frac{i}{k}\right)a + \frac{i}{k}b\right) B_i^k, \quad x \in [a,b], \tag{1}$$

where $(B_0^k, \ldots, B_k^k)$ is the Bernstein basis of degree k on $[a,b]$, *i.e.*, $B_i^k(x) := \binom{k}{i}(\frac{x-a}{b-a})^i(\frac{b-x}{b-a})^{k-i}$. It reproduces any affine function U on $[a,b]$, in the sense that $\mathbb{B}_k U = U$. In [5], S.N. Bernstein proved that, for every function $F \in C^0([a,b])$, $\lim_{k\to+\infty} \|F - \mathbb{B}_k F\|_\infty = 0$. In Section 3, we show how this result extends to the class of Müntz spaces (*i.e.*, spaces spanned by power functions) on a given positive interval $[a,b]$, see Theorem 3.1. Beforehand, in Section 2, we briefly remind the reader how to define operators of the Bernstein-type in Extended Chebyshev spaces.

2. Extended Chebyshev spaces and Bernstein operators

Throughout this section, $[a,b]$ is a fixed non-trivial real interval. For any $n \geqslant 0$, a given $(n+1)$-dimensional space $\mathbb{E} \subset C^n([a,b])$ is said to be an *Extended Chebyshev space* (for short, EC-space) *on* $[a,b]$ when any non-zero element of $\mathbb{E}$ vanishes at most n times on $[a,b]$ counting multiplicities up to $(n+1)$.

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Let $\mathbb{E}$ be an $(n+1)$-dimensional EC-space on $[a,b]$. Then, $\mathbb{E}$ possesses bases $(B_0,\dots,B_n)$ such that, for $i=0,\dots,n$, B_i vanishes exactly i times at a and $(n-i)$ times at b and is positive on $]a,b[$. We say that such a basis is the *Bernstein basis relative to* (a,b) if it additionally satisfies $\sum_{i=0}^{n} B_i = \mathbb{1}$, where $\mathbb{1}$ is the constant function $\mathbb{1}(x)=1$, $x\in[a,b]$. Let us recall that $\mathbb{E}$ possesses a Bernstein basis relative to (a,b) if and only if, firstly it contains constants, and secondly the n-dimensional space $D\mathbb{E} := \{DF := F' \mid F \in \mathbb{E}\}$ is an EC-space on $[a,b]$. Note that the second property is not an automatic consequence of the first one, see [8] and other references therein.

As an instance, given any pairwise distinct $\lambda_0,\dots,\lambda_k$, the so-called *Müntz space* $M(\lambda_0,\dots,\lambda_k)$, spanned over a given positive interval $[a,b]$ (*i.e.*, $a>0$) by the power functions x^{λ_i}, $0\leqslant i\leqslant k$, is a $(k+1)$-dimensional EC-space on $[a,b]$. If $\lambda_0=0$, since $D(M(\lambda_0,\dots,\lambda_k)) = M(\lambda_1-1,\dots,\lambda_k-1)$, the space $M(\lambda_0,\dots,\lambda_k)$ possesses a Bernstein basis relative to (a,b).

For the rest of the section, we assume that $\mathbb{E}\subset C^n([a,b])$ contains constants and that $D\mathbb{E}$ is an (n-dimensional) EC-space on $[a,b]$. We denote by $(B_0,\dots,B_n)$ the Bernstein basis relative to (a,b) in $\mathbb{E}$.

Definition 2.1. A linear operator $\mathbb{B}: C^0([a,b]) \to \mathbb{E}$ is said to be a *Bernstein operator based on* $\mathbb{E}$ when, firstly it is of the form:

$$\mathbb{B}F := \sum_{i=0}^{k} F(\zeta_i)B_i, \quad \text{for some } a=\zeta_0<\zeta_1<\cdots<\zeta_n=b, \tag{2}$$

and secondly it reproduces a two-dimensional EC-space $\mathbb{U}$ on $[a,b]$, in the sense that $\mathbb{B}V=V$ for all $V\in\mathbb{U}$.

Any Bernstein operator $\mathbb{B}$ is positive (*i.e.*, $F\geqslant 0$ implies $\mathbb{B}F\geqslant 0$) and shape preserving due to the properties of Bernstein bases in EC-spaces, see [8]. Note that everything concerning Bernstein-type operators in EC-spaces with no Bernstein bases can be deduced from Bernstein operators as defined above [8,9].

Theorem 2.2. *Given $n\geqslant 2$, let $\mathbb{E}\subset C^n([a,b])$ contain constants. Assume that $D\mathbb{E}$ is an n-dimensional EC-space on $[a,b]$. For a function $U\in\mathbb{E}$, expanded in the Bernstein basis relative to (a,b) as $U:=\sum_{i=0}^{n} u_iB_i$, the following properties are equivalent*:

(i) *$u_0,\dots,u_n$ form a strictly monotonic sequence*;
(ii) *there exists a nested sequence $\mathbb{E}_1\subset\mathbb{E}_2\subset\cdots\subset\mathbb{E}_{n-1}\subset\mathbb{E}_n:=\mathbb{E}$, where $\mathbb{E}_1:=\mathrm{span}(\mathbb{1},U)$ and where, for $i=1,\dots,n-1$, $\mathbb{E}_i$ is an $(i+1)$-dimensional EC-space on $[a,b]$*;
(iii) *there exists a Bernstein operator based on $\mathbb{E}$ which reproduces U.*

In [8] it was proved that there exists a one-to-one correspondence between the set of all Bernstein operators based on $\mathbb{E}$ and the set of all two-dimensional EC-spaces $\mathbb{U}$ they reproduce. In particular, if (i) holds, then the unique Bernstein operator based on $\mathbb{E}$ reproducing U is defined by (2) with:

$$\zeta_i := U^{-1}(u_i), \quad 0\leqslant i\leqslant n. \tag{3}$$

Note that this is meaningful because (i) implies the strict monotonicity of U on $[a,b]$. Condition (ii) of Theorem 2.2 yields the following corollary.

Corollary 2.3. *Given an integer $n\geqslant 1$, consider a nested sequence*:

$$\mathbb{E}_n\subset\mathbb{E}_{n+1}\subset\cdots\subset\mathbb{E}_p\subset\mathbb{E}_{p+1}\subset\cdots, \tag{4}$$

where $\mathbb{E}_n$ contains constants and for any $p\geqslant n$, $D\mathbb{E}_p$ is a p-dimensional EC-space on $[a,b]$. Let $U\in\mathbb{E}_n$ be a non-constant function reproduced by a Bernstein operator $\mathbb{B}_n$ based on $\mathbb{E}_n$. Then, U is also reproduced by a Bernstein operator $\mathbb{B}_p$ based on $\mathbb{E}_p$ for any $p>n$.

Remark 2.4. In the situation described in Corollary 2.3, a natural question arises: given $F\in C^0([a,b])$, does the sequence $\mathbb{B}_kF$, $k\geqslant n$, converges to F in $C^0([a,b])$ equipped with the infinite norm? Obviously, for this to be true for any $F\in C^0([a,b])$, it is necessary that $\bigcup_{k\geqslant n}\mathbb{E}_k$ be dense in $C^0([a,b])$. The example of Müntz spaces proves that this is not always satisfied.

3. Müntz spaces over positive intervals

Throughout this section, we consider a fixed positive interval $[a,b]$, a fixed infinite sequence of real numbers λ_k, $k\geqslant 0$, assumed to satisfy:

$$0=\lambda_0<\lambda_1<\cdots<\lambda_k<\lambda_{k+1}<\cdots, \qquad \lim_{k\to+\infty}\lambda_k=+\infty. \tag{5}$$

We are concerned with the corresponding nested sequence of Müntz spaces:

$$M(\lambda_0)\subset M(\lambda_0,\lambda_1)\subset\cdots\subset M(\lambda_0,\dots,\lambda_k)\subset M(\lambda_0,\dots,\lambda_k,\lambda_{k+1})\subset\cdots \tag{6}$$

Given any $n \geqslant 1$, for each $k \geqslant n$, we can select a Bernstein operator $\mathbb{B}_k$ based on $M(\lambda_0, \dots, \lambda_k)$. Assume the sequence $\mathbb{B}_k$, $k \geqslant n$, to satisfy:

$$\lim_{k \to +\infty} \|F - \mathbb{B}_k F\|_\infty = 0 \quad \text{for any } F \in C^0\big([a,b]\big). \tag{7}$$

Then, the union of all spaces $M(\lambda_0, \dots, \lambda_k)$, $k \geqslant 0$, is dense in $C^0([a,b])$ equipped with the infinite norm. As is well known, this holds if and only if the sequence (5) fulfils the so-called *Müntz density condition* below [4,6]:

$$\sum_{i \geqslant 1} \frac{1}{\lambda_i} = +\infty. \tag{8}$$

As an instance, the Müntz condition (8) is satisfied when $\lambda_k = \ell + 1$ for all $k \geqslant 1$. This case was addressed in [8]. Convergence – in the sense of (7) – was proved there under the assumption that each $\mathbb{B}_k$ reproduced the function x^{λ_1}. This convergence result includes the classical Bernstein operators [5] obtained with $\ell = 0$. Below we extend it to the general interesting situation of sequences of Müntz Bernstein operators $\mathbb{B}_k$ all reproducing the same two-dimensional EC-space (see Remark 2.4).

Theorem 3.1. *Given $n \geqslant 1$, let $\mathbb{E}_1 \subset M(\lambda_0, \dots, \lambda_n)$ be a two-dimensional EC-space reproduced by a Bernstein operator $\mathbb{B}_k$ based on $M(\lambda_0, \dots, \lambda_k)$ for any $k \geqslant n$. Then, if the Müntz density condition (8) holds, the sequence $\mathbb{B}_k$, $k \geqslant n$, converges in the sense of (7).*

Before starting the proof, let us introduce some notations. For $k \geqslant 1$, denote by $(B_{k,0}, \dots, B_{k,k})$ the Bernstein basis relative to (a,b) in the Müntz space $M(\lambda_0, \dots, \lambda_k)$. We consider the functions:

$$U^*(x) = x^{\lambda_1}, \qquad V_p(x) := x^{\lambda_p}, \quad p \geqslant 2,\ x \in [a,b],$$

expanded in the successive Bernstein bases as:

$$U^* = \sum_{i=0}^{k} u^*_{k,i} B_{k,i} \quad \text{for all } k \geqslant 1, \qquad V_p = \sum_{i=0}^{k} v_{p,k,i} B_{k,i} \quad \text{for all } k \geqslant p. \tag{9}$$

With these notations, the key-point to prove Theorem 3.1 is the following lemma, for the proof of which we refer to [1], see also [2].

Lemma 3.2. *Assume that the Müntz density condition (8) holds. Then, we have*:

$$\lim_{k \to +\infty} \max_{0 \leqslant i \leqslant k} \big|\big(u^*_{k,i}\big)^{\frac{\lambda_p}{\lambda_1}} - v_{p,k,i}\big| = 0 \quad \textit{for all } p \geqslant 2. \tag{10}$$

Proof of Theorem 3.1. • Let us start with the simplest example $n = 1$. Then, $\mathbb{E}_1 = \mathrm{span}(\mathbb{1}, U^*)$. For each $k \geqslant 1$, the unique operator $\mathbb{B}^*_k$ which reproduces $\mathbb{E}_1$ is given by:

$$\mathbb{B}^*_k F := \sum_{i=0}^{k} F\big(\zeta^*_{k,i}\big) B_{k,i}, \quad \text{with, for } i = 0, \dots, k,\ \zeta^*_{k,i} := \big(u^*_{k,i}\big)^{\frac{1}{\lambda_1}}. \tag{11}$$

According to Korovkin's theorem for positive linear operators [7], we just have to select a function F so that $\mathbb{1}, U^*, F$ span a three-dimensional EC-space on $[a,b]$ and prove that $\lim_{k \to +\infty} \|F - \mathbb{B}^*_k F\|_\infty = 0$ for this specific F. We can thus choose for instance $F := V_2$. Actually, we will more generally prove the result with $F = V_p$, for any $p \geqslant 2$. Using (9) and (11), we obtain, for any $k \geqslant p$,

$$\big\|\mathbb{B}^*_k V_p - V_p\big\|_\infty = \left\|\sum_{i=0}^{k} \big(V_p\big(\zeta^*_{k,i}\big) - v_{p,k,i}\big) B_{k,i}\right\|_\infty \leqslant \max_{0 \leqslant i \leqslant k} \big|V_p\big(\zeta^*_{k,i}\big) - v_{p,k,i}\big|. \tag{12}$$

On account of (11), Lemma 3.2 yields the expected result:

$$\lim_{k \to +\infty} \big\|\mathbb{B}^*_k V_p - V_p\big\|_\infty = 0 \quad \text{for each } p \geqslant 2.$$

• We now assume that $n > 1$. Select a strictly increasing function $U \in \mathbb{E}_1$. Condition (ii) of Theorem 2.2 enables us to select a function $V \in M(\lambda_0, \dots, \lambda_n)$ so that the functions $\mathbb{1}, U, V$ span a three-dimensional EC-space on $[a,b]$. For any $k \geqslant n$, expand U, V as:

$$U=\sum_{i=0}^{k}u_{k,i}B_{k,i},\qquad V=\sum_{i=0}^{k}v_{k,i}B_{k,i}.$$

We know that, for each $k\geqslant n$, the sequence $(u_{k,0},\dots,u_{k,k})$ is strictly increasing, and that the Bernstein operator $\mathbb{B}_k$ is defined by formula (2) with $\zeta_{k,i}:=U^{-1}(u_{k,i})$ for $i=0,\dots,k$. Via expansions of U and V in the basis $(\mathbb{1},U^*,V_2,\dots,V_n)$ of the Müntz space $M(\lambda_0,\dots,\lambda_n)$, Lemma 3.2 readily proves that:

$$\lim_{k\to+\infty}\max_{0\leqslant i\leqslant k}\left|U\left(\zeta_{k,i}^*\right)-u_{k,i}\right|=0=\lim_{k\to+\infty}\max_{0\leqslant i\leqslant k}\left|V\left(\zeta_{k,i}^*\right)-v_{k,i}\right|.\tag{13}$$

The left part in (13) can be written as $\lim_{k\to+\infty}\max_{0\leqslant i\leqslant k}|U(\zeta_{k,i}^*)-U(\zeta_{k,i})|=0$. On this account, the uniform continuity of the function $V\circ U^{-1}$ and the right part in (13) prove that $\lim_{k\to+\infty}\max_{0\leqslant i\leqslant k}|V(\zeta_{k,i})-v_{k,i}|=0$, thus implying that $\lim_{k\to+\infty}\|\mathbb{B}_kV-V\|_\infty=0$. By Korovkin's theorem, (7) is satisfied. □

Remark 3.3. Given $n\geqslant 2$, one can apply Theorem 3.1 with $\mathbb{E}_1:=\mathrm{span}(\mathbb{1},V_n)=M(\lambda_0,\lambda_n)$, due to the nested sequence of Müntz spaces $M(\lambda_0,\lambda_1,\dots,\lambda_{i-1},\lambda_n)$ for $1\leqslant i\leqslant n$. Note that Theorem 3.1 contains in particular the *Bernstein-type result* expected in [3].

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