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Differential geometry

# Special projective Lichnérowicz–Obata theorem for Randers spaces

*Le théorème projectif resteint de Lichnérowicz–Obata sur les espaces de Randers*

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ABSTRACT

It is proved that either every special projective vector field $V$ on a Randers space $(M, F = \alpha + \beta)$ is a conformal vector field of the Riemannian metric $\alpha^2 - \beta^2$, or $F$ is of isotropic S-curvature. This result is applied to establish a projective Lichnérowicz–Obata-type result on the closed manifolds with generic Randers metrics.


RÉSUMÉ

On prouve que, soit chaque champ projectif de vecteurs sur un espace de Randers $(M, F = \alpha + \beta)$ est conforme à la métrique riemanienne $\alpha^2 - \beta^2$, soit $F$ est à S-courbure isotrope. Ce résultat est appliqué à l'établissement d'un théorème de type de Lichnérowicz–Obata sur les variétés fermées de Randers.


## 1. Introduction

The projective Lichnérowicz–Obata theorem in Riemannian geometry has been recently extended to closed Randers spaces in [4], cf. Corollary 1.5. However, this result seems to be incomplete since, unlike its Riemannian prototype, it does not imply the positivity of the flag curvature of the metric. A suggestion for arriving to the case of positive flag curvature is to consider only a sub-class of projective geometry in order to establish a reduced Lichnérowicz–Obata-type theorem. As it will be presented, the *special projective geometry*—which has been recently discussed in [5–7] for Randers metrics—is a good candidate for such a purpose, since this is an immediate extension of the Riemannian projective geometry.

The results would imply that the special projective Randers geometry may refer to study one of the following cases: (a) conformal transformations of an appropriate Riemannian space, (b) isometries of a Randers space, or (c) Randers spaces of isotropic S-curvature. We prove the result for the pure Randers metrics:

**Theorem 1.1.** *Let us suppose that $(M, F = \alpha + \beta)$ is a Randers space of dimension $n \geqslant 2$. Then, at least one of the following statements holds*:

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(i) *every special projective vector field on* $(M, F)$ *is a conformal vector field of the Riemannian metric* $\alpha^2 - \beta^2$,
(ii) $F$ *is of isotropic S-curvature.*

Theorem 1.1 implies the following result:

**Theorem 1.2.** *Let us suppose that* $(M, F = \alpha + \beta)$ *is a closed and connected Randers space of dimension* $n \geqslant 2$ *and* $V$ *is a special projective vector field of* $F$*. Then, at least one of the following statements holds*:

(i) $V$ *is a conformal vector field for the Riemannian metric* $\alpha^2 - \beta^2$,
(ii) *there is a Randers metric* $\hat{F}$ *such that* $V$ *is a Killing vector field for* $\hat{F}$,
(iii) *after an appropriate rescaling,* $F$ *is of the following local form*:

$$F(x, y) = \frac{\sqrt{|y|^2 + |x|^2|y|^2 - \langle x, y\rangle^2}}{1 + |x|^2} - \frac{f_{x^k} y^k}{\sqrt{1 - f^2(x)}}, \quad y \in T_xM \cong \mathbb{R}^n, \tag{1}$$

*where* $f$ *is an eigenfunction of the standard Laplacian satisfying* $\Delta f = nf$ *and* $\max_{x\in M} |f| < 1$. *In particular,* $F$ *is of positive flag curvature.*

All manifolds are assumed to be smooth and connected, the natural coordinates on the tangent manifold $TM$ are denoted by $(x^i, y^i)$ and the derivations with respect to $x^k$ and $y^k$ are denoted by the subscripts $_{x^k}$ and $_{y^k}$, respectively. Moreover, we deal with pure and positive definite Randers metrics.

## 2. Special projective Finsler geometry

Two Finsler metrics $F$ and $\tilde{F}$ on $M$ are said to be *projectively equivalent* if they have the same forward geodesics. A Finsler metric $F$ is said to be *locally projectively flat* if, at any point $x \in M$, there is a neighborhood $U$ such that $F$ and the Euclidean metric are projectively equivalent on $U$. Given a Finsler space $(M, F)$, a diffeomorphism $\phi : M \longrightarrow M$ is called a *projective transformation* if $F$ and $\phi^*F$ are projectively equivalent.

Suppose that $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form defined on $M$ such that $\|\beta\|_x := \sup_{y\in T_xM\setminus\{0\}} \beta(y)/\alpha(y) < 1$. Then the function $F = \alpha + \beta$ is a Finsler metric on $M$, which is called a *Randers metric*. The geodesic spray coefficients of $\alpha$ and $F$ are denoted respectively by the $G^i_\alpha$ and $G^i$, and the Levi-Civita connection of $\alpha$ is denoted by $\nabla$. The covariant derivation of $\beta$ is given by $(\nabla_j b_i)\,dx^j := db_i - b_j \theta_i{}^j$, where $\theta_i{}^j := \tilde{\Gamma}^j_{ik}\,dx^k$ denote the associated connection forms. Let us stipulate the following conventions: $r_{ij} := \frac{1}{2}(\nabla_j b_i + \nabla_i b_j)$, $s_{ij} := \frac{1}{2}(\nabla_j b_i - \nabla_i b_j)$, $s^i{}_j := a^{ih} s_{hj}$, $s_j := b_i s^i{}_j$ and $e_{ij} := r_{ij} + b_i s_j + b_j s_i$, $e_{00} := e_{ij} y^i y^j$, $s_0 := s_i y^i$ and $s^i{}_0 := s^i{}_j y^j$. Then the geodesic spray coefficients $G^i$ of $F$ are of the following form:

$$G^i = G^i_\alpha + \left(\frac{e_{00}}{2F} - s_0\right) y^i + \alpha s^i{}_0. \tag{2}$$

It is well known that a Randers metric $F = \alpha + \beta$ on $M$ is locally projectively flat if and only if $\alpha$ is of constant sectional curvature and if $\beta$ is closed. The locally projectively flat Randers metrics with isotropic S-curvature has been characterized by Chen, Mo and Shen in [2], cf. Theorem 1.3 and Theorem 1.4.

A projective transformation $\phi : M \longrightarrow M$ is said to be *special* if it preserves the E-curvature; in this case, $\phi$ changes the geodesic spray coefficients as $\tilde{G}^i(x, y) = G^i(x, y) + P(x, y) y^i$, where $P = P_i(x) y^i$. The complete lift of any vector field $V$ on $M$ is given by $\hat{V} = V^i \frac{\partial}{\partial x^i} + y^k \frac{\partial V^i}{\partial x^k} \frac{\partial}{\partial y^i}$. The Lie derivative operator with respect to the vector field $V$ is denoted by $\mathcal{L}_{\hat{V}}$. It is well known that, $\mathcal{L}_{\hat{V}} y^i = 0$, $\mathcal{L}_{\hat{V}}\,dx^i = 0$ and the differential operators $\mathcal{L}_{\hat{V}}$, $\frac{\partial}{\partial x^i}$, the exterior differential operator d and $\frac{\partial}{\partial y^i}$ commute within any natural coordinates system on tangent manifold. The vector field $V$ is called a *projective vector field*, if there is a function $P$ on $TM_0$, called the *projective factor*, such that $\mathcal{L}_{\hat{V}} G^i = P y^i$, see [1]. In this case, given any appropriate $t$, the local flow $\{\phi_t\}$ associated with $V$ is a projective transformation. A projective vector field $V$ is said to be *special* if the projective factor $P(x, y)$ is lift of a 1-form on $M$, i.e. $P(x, y) = P_i(x) y^i$. Notice that, on the Riemannian spaces, given any projective vector field $V$, the projective factor $P(x, y)$ is linear with respect to $y$, while this property is a non-Riemannian feature in a Finslerian background. The projective vector fields have several characterizations in the contexts, see Ref. [1] for some such results. The following characterization is useful in the sequel:

**Theorem 2.1.** *(See [5–7].) A vector field* $V$ *is projective on a Randers space* $(M, F = \alpha + \beta)$ *if and only if* $V$ *is projective on* $(M, \alpha)$ *and* $\mathcal{L}_{\hat{V}}(\alpha s^i{}_0) = 0$.

Given any vector field $V$, let us stipulate the notation $t_{00} = \mathcal{L}_{\hat{V}} \alpha^2$. Now, we prove the following characterization of special projective vector fields on Randers spaces:

**Lemma 2.2.** *A vector field $V$ on a Randers space $(M, F = \alpha + \beta)$ is special projective if and only if there is a 1-form $P = P_i(x)y^i$ such that the following equations hold*:

(1) $8\alpha^2\beta\mathcal{L}_{\hat{V}}G^i_\alpha + (2\alpha^2\mathcal{L}_{\hat{V}}e_{00} - e_{00}t_{00} - 8\alpha^2\beta(\mathcal{L}_{\hat{V}}s_0 - P))y^i = 0$,
(2) $4(\alpha^2 + \beta^2)\mathcal{L}_{\hat{V}}G^i_\alpha + (2\beta\mathcal{L}_{\hat{V}}e_{00} - 2e_{00}\mathcal{L}_{\hat{V}}\beta - 4(\alpha^2 + \beta^2)(\mathcal{L}_{\hat{V}}s_0 - P))y^i = 0$.

**Proof.** A vector field $V$ on $(M, F)$ is special projective if and only if there is a 1-form $P = P_i(x)y^i$ on $M$ such that $\mathcal{L}_{\hat{V}}G^i = Py^i$. By (2) and Theorem 2.1, this is equivalent to:

$$\mathcal{L}_{\hat{V}}\left(G^i_\alpha + \left(\frac{e_{00}}{2F} - s_0\right)y^i\right) = Py^i. \tag{3}$$

After expanding the terms, Eq. (3) is equivalent to the identities below:

$$\begin{aligned}
0 &= \mathcal{L}_{\hat{V}}\left(G^i_\alpha + \left(\frac{e_{00}}{2F} - s_0\right)y^i\right) - Py^i = \mathcal{L}_{\hat{V}}G^i_\alpha + \mathcal{L}_{\hat{V}}\frac{e_{00}}{2F}y^i - \mathcal{L}_{\hat{V}}s_0y^i - Py^i \\
&= \mathcal{L}_{\hat{V}}G^i_\alpha + \frac{\mathcal{L}_{\hat{V}}e_{00}}{2F}y^i - \frac{e_{00}\mathcal{L}_{\hat{V}}F}{2F^2}y^i - \mathcal{L}_{\hat{V}}s_0y^i - Py^i \\
&= \mathcal{L}_{\hat{V}}G^i_\alpha + \frac{\mathcal{L}_{\hat{V}}e_{00}}{2F}y^i - e_{00}\frac{\frac{t_{00}}{2\alpha} + \mathcal{L}_{\hat{V}}\beta}{2F^2}y^i - \mathcal{L}_{\hat{V}}s_0y^i - Py^i = \frac{1}{4\alpha F^2}\{Rat^i + \alpha Irrat^i\},
\end{aligned}$$

where,

$$Rat^i = 8\alpha^2\beta\mathcal{L}_{\hat{V}}G^i_\alpha + \big(2\alpha^2\mathcal{L}_{\hat{V}}e_{00} - e_{00}t_{00} - 8\alpha^2\beta(\mathcal{L}_{\hat{V}}s_0 - P)\big)y^i, \tag{4}$$

$$Irrat^i = 4\big(\alpha^2 + \beta^2\big)\mathcal{L}_{\hat{V}}G^i_\alpha + \big(2\beta\mathcal{L}_{\hat{V}}e_{00} - 2e_{00}\mathcal{L}_{\hat{V}}\beta - 4\big(\alpha^2 + \beta^2\big)(\mathcal{L}_{\hat{V}}s_0 - P)\big)y^i. \tag{5}$$

Hence, $V$ is a special projective vector field if and only if $Rat^i = 0$ and $Irrat^i = 0$, for every $i = 1, \ldots, n$. This completes the proof. □

## 3. Proof of main theorems

**Proof of Theorem 1.1.** Let us suppose that $V$ is an arbitrary special projective vector field on $(M, F = \alpha + \beta)$. From Lemma 2.2, there is a 1-form $P = P_i(x)y^i$ on $M$ such that $Rat^i = 0$ and $Irrat^i = 0$, for any index $i$; Notice that, $Rat^i$ and $Irrat^i$ are given in (4) and (5). Now, it follows that:

$$\begin{aligned}
0 &= Rat^i - \beta Irrat^i \\
&= 4\big(\alpha^2 - \beta^2\big)\beta\mathcal{L}_{\hat{V}}G^i_\alpha + 2\big(\alpha^2 - \beta^2\big)\mathcal{L}_{\hat{V}}e_{00}y^i - e_{00}\mathcal{L}_{\hat{V}}\big(\alpha^2 - \beta^2\big)y^i - 4\beta\big(\alpha^2 - \beta^2\big)(\mathcal{L}_{\hat{V}}s_0 - P)y^i \\
&= \big(\alpha^2 - \beta^2\big)Q^i - e_{00}\mathcal{L}_{\hat{V}}\big(\alpha^2 - \beta^2\big)y^i \quad (i = 1, \ldots, n),
\end{aligned}$$

where, $Q^i = \{4\beta\mathcal{L}_{\hat{V}}G^i_\alpha + 2\mathcal{L}_{\hat{V}}e_{00}y^i - 4\beta(\mathcal{L}_{\hat{V}}s_0 - P)y^i\}$. Given any point $x \in M$, the irreducible polynomial $(\alpha^2 - \beta^2) \in \mathbb{R}[y^1, \ldots, y^n]$ divides the polynomials $e_{00}\mathcal{L}_{\hat{V}}(\alpha^2 - \beta^2)y^i$ $(i = 1, \ldots, n)$. Notice that $(\alpha^2 - \beta^2)$ cannot divide $y^i$ for any index $i$. Given any special projective vector field $V$, if $(\alpha^2 - \beta^2)$ divides $\mathcal{L}_{\hat{V}}(\alpha^2 - \beta^2)$, then it follows that $V$ is a conformal vector field of the Riemannian metric $(\alpha^2 - \beta^2)$ and this proves (i) in Theorem 1.1. Otherwise, $(\alpha^2 - \beta^2)$ divides $e_{00}$; in this case, $F$ is of isotropic S-curvature, cf. [3], and this proves (ii). □

**Proof of Theorem 1.2.** Suppose that $V$ is a special projective vector field on $(M, F = \alpha + \beta)$ which is not a conformal vector field of the Riemannian metric $\alpha^2 - \beta^2$. By Theorem 1.1, $F$ is of isotropic S-curvature. Moreover, by a result in [4], cf. Corollary 1.5, there is a Randers metric $\hat{F}$ such that $V$ is either a Killing vector field of $\hat{F}$ or $F$ is locally projectively flat and $\alpha$ has positive constant sectional curvature. In the latter case, by a result in [2], cf. the case (c) in Theorem 1.4, after an appropriate rescaling, $F$ is locally isometric to the Randers metric given by $F(x, y) = \alpha(x, y) - f_{x^k}y^k/\sqrt{1 - f(x)^2}$, where $f$ is an eigenfunction of the standard Laplacian corresponding to the eigenvalue $\lambda = n$ with $\max_{x\in M}|f(x)| < 1$. Moreover, the flag curvature and the S-curvature of $F$ are of the following forms:

$$\mathbf{K}(x, y) = \frac{1}{4} + \frac{3F(x, -y)}{4(1 - f(x)^2)F(x, y)}, \qquad \mathbf{S}(x, y) = (n + 1)\frac{f(x)}{2\sqrt{1 - f(x)^2}}F(x, y).$$

It can be checked now that we have $\mathbf{K} > 0$. □

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