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Reconstruction of extended sources with small supports in the elliptic equation $\Delta u + \mu u = F$ from a single Cauchy data

Reconstruction de sources dont le support est de petite taille, dans l'équation elliptique $\Delta u + \mu u = F$, à partir d'une seule donnée de Cauchy

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A B S T R A C T

This Note focuses on an algebraic reconstruction method allowing to solve an inverse source problem in the elliptic equation $\Delta u + \mu u = F$ from a single Cauchy data. The source term F is a distributed function having compact support within a finite number of small subdomains.

R É S U M É

Cette Note porte sur une méthode algébrique permettant de résoudre un problème inverse de sources dans l'équation elliptique $\Delta u + \mu u = F$ à partir d'une seule donnée de Cauchy. Le terme source F est une fonction distribuée à support compact contenu dans un ensemble fini de sous-domaines de petites tailles.

Version française abrégée

Soit Ω un domaine borné de $\mathbb{R}^3$ de frontière Γ supposée suffisamment régulière. On considère dans cette Note le problème de détermination du terme source F dans le problème elliptique :

$$\Delta u + \mu u = F \quad \text{dans } \Omega,$$

à partir d'une seule donnée de Cauchy $(f, g) := (u_{|\Gamma}, \frac{\partial u}{\partial n}_{|\Gamma})$. Ici, μ est un nombre réel donné et la source F est supposée être distribuée a support compact contenu dans un ensemble fini de sous-domaines de petites tailles, à savoir $F = \sum_{j=1}^{N} q_j \chi_{D_j}$ avec $D_j = S_j + \varepsilon B_j$, où $B_j \subset \mathbb{R}^3$ est un domaine borné contenant l'origine. Les points $S_j \in \Omega$ sont supposés mutuellement distincts et ε est un réel positif plus petit que 1. L'objet de cette Note est d'établir un résultat caractérisant le nombre de sources et leurs localités. Pour la simplicité de la présentation, nous supposons dans le cadre de cette Note les intensités constantes. Un résultat plus général dans le cas de fonctions C^n sera présenté dans un travail à venir [1].

Théorème 0.1. *Soient n un entier naturel donné et $J = (n+1)N$.*

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(i) *Soit $K \in \mathbb{N}$ tel que $K \geqslant J$, alors la matrice de Hankel $H_{K,n}$ définie dans (5) est de rang $(n+1)N$.*
(ii) *La matrice Companion B_n, définie dans (6), admet N valeurs propres de multiplicités $n+1$. Ces valeurs propres sont les projections Q_j des points S_j sur le plan complexe xy.*

1. Introduction and main result

Inverse problems (IP) are very important in science, engineering, and bioengineering. Among these, inverse source problems (ISP) have attracted great attention from many researchers over the recent years because of their applications to many practical examples, particularly in biomedical imaging techniques as the so-called inverse electroencephalography/magnetoencephalography (EEG/MEG) problems [7,10], pollution in the environment [5], photo- and thermo-acoustic tomography [12], optimal tomography [2], and bioluminescence tomography [14].

In this Note, we consider the problem of determining a source F in the following elliptic equation:

$$\Delta u + \mu u = F \quad \text{in } \Omega, \tag{1}$$

from a single Cauchy data $(f, g) := (u_{|\Gamma}, \frac{\partial u}{\partial n}_{|\Gamma})$ prescribed on the boundary Γ of Ω, where $\Omega \subset \mathbb{R}^3$ is an open bounded domain with a sufficiently regular boundary Γ. Here μ is a fixed real number assumed to be known.

One of the difficulties of the inverse source problem from boundary measurements concerns the non-uniqueness of the source, for example due to the possible existence of non-radiating sources. Also, it is obvious that, if we add to the solution u of (1) any function or distribution v with support in Ω, we get a solution of the same equation with a (possibly) different RHS source F and the same boundary data. Thus, a general source F cannot be identified from boundary measurements when no a priori information is available.

In this Note we assume that F represents sources having compact support within a finite number of subdomains, namely:

$$F = \sum_{j=1}^{N} q_j \chi_{D_j} \quad \text{with } D_j = S_j + \varepsilon B_j$$

where $S_j = (x_j, y_j, z_j)$ and $B_j \subset \mathbb{R}^3$ is a bounded domain containing the origin. The points $S_j \in \Omega$ are assumed to be mutually distinct and ε is a positive real number less than 1. For the sake of simplicity, we assume in this Note the intensities q_j to be constant. The case of varying functions will be presented in a future work [1].

The inverse source problem we are concerned with here consists of determining the number N and the locations S_j from a single Cauchy data belonging to the space $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$.

Two applications of the inverse problem we consider here can be the Helmholtz equation in an interior domain (corresponding to μ positive [6]) and the bioluminescence tomography (corresponding to μ negative), but they are not the only ones. For the convenience for the reader, we will give a brief description of the bioluminescence tomography problem. In fact, the bioluminescence tomography consists in determining an internal bioluminescent source distribution generated by luciferase inducted by reporter genes, from external optical measurements [14]. The mathematical model underlying is given by the radiative transfer equation corresponding to light migration in a random medium. Solving the inverse source problem in the radiative transfer equation is a hard problem. However, since the mean-free path of the particle is between 0.005 and 0.01 mm in biological tissues, which is very small compared to a typical object in this context, then the predominant phenomenon is scattering instead of transport, which leads to a diffusion equation satisfied by the average photon flux in all directions. This has been widely used in optical tomography [2]. Finally, as the internal bioluminescence distribution inducted by reporter genes is relatively stable, one can see the bioluminescence tomography as an inverse source problem for an elliptic equation, see [14] and the references therein.

Before formulating our main result, we introduce some notation and specify additional information.

First, we introduce the space of the homogeneous Helmholtz equation in Ω:

$$\mathcal{H}_\mu = \{ v \in H^1(\Omega) \colon \Delta v + \mu v = 0 \}$$

and define the operator $\mathcal{R}$ as follows:

$$\mathcal{R}(v, f, g) = \int_{\Gamma} \left(g v - f \frac{\partial v}{\partial n} \right) \mathrm{d}s \quad \text{for all } v \in \mathcal{H}_\mu.$$

Multiplying Eq. (1) by v, an element of $\mathcal{H}_\mu$, integrating by parts and then using the Green formula gives:

$$\mathcal{R}(v, f, g) = \sum_{j=1}^{N} \int_{D_j} q_j v(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z, \quad \text{for all } v \in \mathcal{H}_\mu.$$

Using the change of variables $(x, y, z) = S_j + \varepsilon t$ with $t = (t_1, t_2, t_3)$, one obtains:

$$\mathcal{R}(v, f, g) = \sum_{j=1}^{N} \varepsilon^3 q_j \int_{B_j} v(S_j + \varepsilon t)\,\mathrm{d}t, \quad \text{for all } v \in \mathcal{H}_\mu. \tag{2}$$

Otherwise, one has:

$$v(S_j + \varepsilon t) = \sum_{\alpha=0}^{n} \varepsilon^\alpha \frac{t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3}}{\alpha_1!\alpha_2!\alpha_3!} \frac{\partial^\alpha}{\partial_x^{\alpha_1} \partial_y^{\alpha_2} \partial_z^{\alpha_3}} v(S_j) + O\left(\varepsilon^{n+1}\right)$$

where $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, with $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$. Thus, formula (2) can be rewritten as:

$$\mathcal{R}(v, f, g) = \sum_{j=1}^{N} \sum_{\alpha=0}^{n} q_j \lambda_{j,(\alpha_1,\alpha_2,\alpha_3)} \frac{\partial^\alpha}{\partial_x^{\alpha_1} \partial_y^{\alpha_2} \partial_z^{\alpha_3}} v(S_j) + O\left(\varepsilon^{n+4}\right) \quad \text{for all } v \in \mathcal{H}_\mu, \tag{3}$$

where he have noted $\lambda_{j,(\alpha_1,\alpha_2,\alpha_3)} = \varepsilon^{\alpha+3} \int_{B_j} \frac{t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3}}{\alpha_1!\alpha_2!\alpha_3!}\,\mathrm{d}t$.

Remark 1.1. Observe that $\lambda_{j,(0,0,0)}$ corresponds to the volume of domains D_j and $\lambda_{j,(1,0,0)}$, $\lambda_{j,(0,1,0)}$, $\lambda_{j,(0,0,1)}$ correspond to their moments.

Now, the question is how to choose special functions v in $\mathcal{H}_\mu$ allowing us to determine the unknown quantities N, S_j and $\lambda_{j,(\alpha_1,\alpha_2,\alpha_3)}$. First, we observe that, for all $m \in \mathbb{N}$, the functions:

$$v_m(x, y, z) = (x + \mathrm{i}y)^m \mathrm{e}^{kz} \quad \text{with } k^2 = -\mu$$

belong to the space $\mathcal{H}_\mu$. Then, we set:

$$c_m = \mathcal{R}(v_m, f, g) \quad \text{modulo} \quad O\left(\varepsilon^{n+4}\right), \tag{4}$$

define the complex Hankel matrix:

$$H_{K,n} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{K-1} \\ c_1 & c_2 & \cdots & c_K \\ \vdots & \vdots & \vdots & \vdots \\ c_{K-1} & c_K & \cdots & c_{2K-2} \end{pmatrix} \quad \text{for } K \in \mathbb{N}^* \tag{5}$$

and introduce the Companion matrix:

$$B_n = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ b_0 & b_1 & \cdots & \cdots & b_{J-1} \end{pmatrix} \quad \text{for } J = (n+1)N, \tag{6}$$

where the vector $b = (b_0, \ldots, b_{J-1})^t$ is a solution of the linear system $H_J b = \xi_J$ with $\xi_J = (c_J, \ldots, c_{2J-1})^t$.

Now we have the following result.

Theorem 1.1. *Let n be a non-negative integer and $J = (n+1)N$.*

(i) *Assume that we know an upper bound K for the number J, then the rank of the Hankel matrix $H_{K,n}$, defined in (5), is $(n+1)N$.*
(ii) *The Companion matrix B_n, defined in (6), admits N eigenvalues of multiplicity $n+1$. These eigenvalues are the projections Q_j of points S_j onto the xy complex plane.*

The present result is based on the algebraic method initially proposed in [4], where the authors have considered a pointwise inverse source problem for the Poisson equation in 2D or 3D. It has been, originally, proposed by A. El Badia and T. Ha-Duong in the case when the multipoles are either monopoles or dipoles, extended by Nara in [11] for a combination of monopoles and dipoles having the same locations and generalized in [3] to a combination of monopoles and dipoles with distinct locations. The methods used in [11,3] are based on calculations of determinants, which are long and tedious. In contrast, our method is more general and simpler. Moreover, a generalization to the combination of multipolar sources with distinct locations as well as numerical experiments are in progress in [1]. Concerning our problem, one can also mention the interesting and relevant paper [9] on the reconstruction of extended sources for the 2D Helmholtz equation. See also the references therein, notably [8].

Remark 1.2. We note that, in practice, for given positive constant $\varepsilon < 1$, we choose the integer n such that ε^{n+4} is small enough. Then, we estimate the coefficients c_m, defined in (4), by $\mathcal{R}(v_m, f, g)$. This introduces an accuracy error $O(\varepsilon^{n+4})$ in our identification algorithm, precisely, in determining of the rank of the Hankel matrix $H_{K,n}$ and the eigenvalues of the Companion matrix B_n (see [13, pp. 321–322] for estimating result on SVD). Therefore, through Theorem 1.1 we can find, modulo a small error, the number of the sources and the projections (onto the xy complex plane) of their positions. To determine the positions of point sources, we repeat the same algorithm by making projections onto the xz and yz complex planes. It is also possible to find, in the case $q_j = 1$, the coefficients $\lambda_{j,(\alpha_1,\alpha_2,\alpha_3)}$ and in particular the volume and the moments of domains D_j; see [1] for more details.

Sketch of the proof of Theorem 1.1. Proof of (i). First, from formula (3), we obtain, for all $m \in \mathbb{N}$, the following relationships, which are behind our identification method:

$$c_m = \sum_{j=1}^{N}\sum_{\alpha=0}^{n} \eta_{j,(\alpha_1,\alpha_2,\alpha_3)}(\alpha_1+\alpha_2)!\binom{m}{\alpha_1+\alpha_2} Q_j^{m-(\alpha_1+\alpha_2)}$$

where $Q_j = x_j + iy_j$, $\eta_{j,(\alpha_1,\alpha_2,\alpha_3)} = (i)^{\alpha_2}(k)^{\alpha_3} q_j \lambda_{j,(\alpha_1,\alpha_2,\alpha_3)} e^{kz_j}$ and $\binom{k}{j} = \frac{k!}{j!(k-j)!}$. Thus, that we can rewrite:

$$c_m = \sum_{j=1}^{N}\sum_{\beta=0}^{n} \nu_j^{\beta}\binom{m}{\beta} Q_j^{m-\beta} \quad \text{for all } m \in \mathbb{N} \tag{7}$$

where, for $\beta = 0,\dots,n$, we have noted: $\nu_j^{\beta} = \beta! \sum_{\alpha_3=0}^{n-\beta} \sum_{\alpha_1+\alpha_2=\beta} \eta_{j,(\alpha_1,\alpha_2,\alpha_3)}$.

Now, let us define, for all $m \in \mathbb{N}$, the complex matrices A_m of size $K \times (n+1)N$ as:

$$A_m = \left(U_m^0, \dots, U_m^n\right)$$

where, for $j = 0,\dots,K$, U_m^j are the Confluent $K \times N$ Vandermonde matrices:

$$U_m^j = \begin{pmatrix} \binom{m}{j} Q_1^{m-j} & \cdots & \binom{m}{j} Q_N^{m-j} \\ \binom{m+1}{j} Q_1^{m-j+1} & \cdots & \binom{m+1}{j} Q_N^{m-j+1} \\ \vdots & \ddots & \vdots \\ \binom{K+m-1}{j} Q_1^{m-j+K-1} & \cdots & \binom{K+m-1}{j} Q_N^{m-j+K-1} \end{pmatrix}.$$

Then, using the algebraic equations (7), we can see that $H_{K,n} = A_0[\Lambda, T\Lambda, \dots, T^{K-1}\Lambda]$, where T is the following upper triangular complex matrix:

$$T = \begin{pmatrix} D_Q & I & \cdots & 0 \\ \ddots & \ddots & \ddots & \vdots \\ \cdots & 0 & D_Q & I \\ \cdots & \cdots & 0 & D_Q \end{pmatrix}, \quad \text{with } D_Q = \operatorname{diag}(Q_1,\dots,Q_N) \text{ and } I = \operatorname{diag}(1,\dots,1),$$

and Λ is the complex vector $\Lambda = (\bar{\nu}^0,\dots,\bar{\nu}^n)^t$, where $\bar{\nu}^\ell = (\nu_1^\ell,\dots,\nu_N^\ell)$. Therefore, we can verify that $H_{K,n} = A_0 \bar{I}(A_0)^t$, where $(A_0)^t$ denotes the matrix transpose of A_0 and:

$$\bar{I} = \begin{pmatrix} \nu^0 & \nu^1 & \cdots & \nu^K \\ \vdots & \vdots & \iddots & \vdots \\ \nu^{K-1} & \nu^K & \cdots & 0 \\ \nu^K & 0 & \cdots & 0 \end{pmatrix}, \quad \text{with } \nu^j = \operatorname{diag}\left(\nu_1^j,\dots,\nu_N^j\right),$$

from which the result follows.

Proof of (ii). It suffices to observe that for $\xi_m = (c_m,\dots,c_{K+m-1})^t$, we have $\xi_{m+1} = A_0 T (A_0)^{-1}\xi_m$, $\forall m \in \mathbb{N}$ and then $B_n = A_0 T (A_0)^{-1}$. That ends the proof of the theorem. □

Remark 1.3. When $\mu \neq 0$, our method is valid only in $\mathbb{R}^3$, while for $\mu = 0$, it is applicable in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

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