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Algebraic Geometry

# Surfaces in $\mathbb{P}^4$ whose 4-secant lines do not sweep out a hypersurface ☆

## *Surfaces de $\mathbb{P}^4$ dont les droites quadrisécantes ne couvrent pas une hypersurface*

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ABSTRACT

We prove that a smooth surface in $\mathbb{P}^4$ whose 4-secant lines do not sweep out a hypersurface of $\mathbb{P}^4$ either lies on a pencil of cubic hypersurfaces, or else is linked to a Veronese surface by the complete intersection of a cubic and a quartic hypersurface.



RÉSUMÉ

Nous montrons qu'une surface lisse dans $\mathbb{P}^4$ dont les droites quadrisécantes ne couvrent pas une hypersurface de $\mathbb{P}^4$ est, soit contenue dans un pinceau de cubiques, soit liée à une surface de Veronese via l'intersection complète d'une cubique et d'une quartique.



## 1. Introduction

Let $X \subset \mathbb{P}^4$ be a smooth complex projective surface. A line $L \subset \mathbb{P}^4$ is said to be $k$-secant to $X$ if $X \cap L$ is a finite scheme of length at least $k$. While the 2-secant lines of $X$ fill up $\mathbb{P}^4$ unless $X$ lies on a hyperplane, Aure [2] characterized the elliptic quintic scrolls — refining earlier work of Severi in his celebrated paper [19] — as the only smooth surfaces not lying on a quadric hypersurface whose 3-secant lines do not fill up $\mathbb{P}^4$, as conjectured by Peskine. On the other hand, Ran's generalization of the classical Trisecant Lemma [18] shows that the 4-secant lines of $X$ never fill up $\mathbb{P}^4$. In this case, $X$ is expected to have a 2-dimensional family of 4-secant lines sweeping out a hypersurface of $\mathbb{P}^4$. Therefore, it is natural to ask whether there are any exceptions to this expected behavior. Of course, the 4-secant lines of a surface lying on a pencil of cubic hypersurfaces do not swept out a hypersurface, so in the spirit of Aure's work we show that a smooth surface whose 4-secant lines do not sweep out a hypersurface of $\mathbb{P}^4$ either lies on a pencil of cubic hypersurfaces, or else is linked to a Veronese surface by the complete intersection of a cubic and a quartic hypersurface. We would like to emphasize the analogy with Aure's result, which in fact can be rephrased by saying that a smooth surface whose 3-secant lines do not fill up $\mathbb{P}^4$ either lies on a quadric hypersurface, or else is linked to a Veronese surface by the complete intersection of two cubic hypersurfaces.

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In higher dimensions, Ran [17] proved — under an extra assumption that is satisfied as soon as $n \geqslant 4$ — that the $(n+1)$-secant lines of a smooth $n$-dimensional subvariety $X \subset \mathbb{P}^{n+2}$ fill up the ambient space if $X$ does not lie on a hypersurface of degree $n$. On the other hand, Mezzetti [15, Theorem 0.2] and Kwak [10, Theorem 3.4(b)] obtained some partial results that suggest that the same could be true in the case $n = 3$. In view of [18] and our result, it would be interesting to study also the smooth $n$-dimensional subvarieties of $\mathbb{P}^{n+2}$ whose $(n+2)$-secant lines do not sweep out a hypersurface of $\mathbb{P}^{n+2}$ (cf. [10, Open questions 4.7]), but we will not address this problem here.

Going back to the case $n = 2$, there are several ways to proceed. In this paper, we give a short proof based on Le Barz's formula [13] for the 4-secant cycle of $X \subset \mathbb{P}^4$, that allows us to express the Euler characteristic $\chi(\mathcal{O}_X)$ in terms of the degree $d$ and the sectional genus $g$ of $X$. Now we come to the key fact of the proof: as the 4-secant lines of $X$ do not sweep out a hypersurface of $\mathbb{P}^4$, the inner projection from a general point of $X$ into $\mathbb{P}^3$ does not have any triple point, and hence we can express $g$ in terms of $d$ thanks to Kleiman's triple-point formula. To conclude the proof, Halphen's bound yields a short list of admissible pairs $(d, g)$ for which the corresponding surface is well known.

We point out that Bauer [3] classified — in response to a conjecture of Van de Ven — the smooth surfaces $X \subset \mathbb{P}^5$ whose 3-secant lines do not sweep out a 3-dimensional subvariety of $\mathbb{P}^5$ in a similar way, that is, using Le Barz's formula for the 3-secant cycle of $X \subset \mathbb{P}^5$ and noting that the inner projection from a general point of $X$ into $\mathbb{P}^4$ does not have any double point.

Finally, we mention that smooth surfaces with no 4-secant lines were classified first by Bertolini and Turrini [4], as explained in Remark 4.

## 2. Proof

We work over the field of complex numbers.

**Theorem.** *Let $X \subset \mathbb{P}^4$ be a smooth surface whose 4-secant lines do not sweep out a hypersurface of $\mathbb{P}^4$. Then either $X$ lies on a pencil of cubic hypersurfaces, or else $X$ is linked to a Veronese surface by the complete intersection of a cubic and a quartic hypersurface.*

The proof is based on the following formula. Let $d$ denote the degree of $X$, let $g := g(C)$ denote the genus of a general hyperplane section $C$ of $X$, and let $\chi := \chi(\mathcal{O}_X)$ denote the Euler characteristic of $X$.

**Le Barz's formula.** *(See [13] and [14].) The number $N_4$ of 4-secant lines of a smooth surface $X \subset \mathbb{P}^4$ meeting a general line, if finite, is:*

$$N_4 = \frac{1}{8}\bigl(d^4 - 10d^3 + d^2(35 - 8g) + 2d(28g - 33) + 4\bigl(g^2 - 25g + 24\bigr) + 8\chi(2d - 9)\bigr).$$

The key fact of the proof is the following:

**Lemma.** *If the 4-secant lines of a smooth surface $X \subset \mathbb{P}^4$ do not sweep out a hypersurface and $X$ is not a scroll (i.e. $X$ is not covered by lines), then*

$$g = \frac{1}{6}\bigl(9d - 33 \pm \sqrt{\Delta(d)}\bigr),$$

*where* $\Delta(d) := 3d^4 - 72d^3 + 636d^2 - 2448d + 3465$.

**Proof.** Let $x \in X$ be a general point, and let $\mathrm{Bl}_x(X)$ denote the blowing-up of $X$ at $x$. It follows from the hypotheses that the map $f : \mathrm{Bl}_x(X) \to \mathbb{P}^3$ induced by the inner projection $\pi_x : X \dashrightarrow \mathbb{P}^3$ is finite and does not have any triple point. Hence we apply Kleiman's triple-point formula to $f$ (see [9] for the general picture; see also [13] for our particular situation), so

$$\chi = \frac{1}{12}\bigl(-d^3 + 9d^2 - 2d(16 - 3g) - 12(2g - 5)\bigr)$$

(cf. [6, Proposition 3.2]) and the statement follows from Le Barz's formula since $N_4 = 0$. □

**Remark 1.** On the other hand, if $X \subset \mathbb{P}^4$ is a scroll then there exists a smooth irreducible curve $B \subset \mathbb{G}(1,4)$ of genus $g(B)$ such that $X \cong \mathbb{P}(E)$, where $E$ denotes the rank-2 universal bundle on $\mathbb{G}(1,4)$ restricted to $B$. Then $g = g(B)$, $\chi = 1 - g$, $K^2 = 8 - 8g$ and hence $g = (d^2 - 5d + 6)/6$ by the well-known double-point formula

$$d^2 = 5d + 10(g - 1) + 2K^2 - 12\chi.$$

Therefore, if $N_4 = 0$ then $(d, g) \in \{(2,0), (3,0), (5,1)\}$ (cf. [11] and [1]).

**Proof of the theorem.** If $X \subset \mathbb{P}^4$ is a scroll then $(d, g) \in \{(2, 0), (3, 0), (5, 1)\}$ by Remark 1. Otherwise, it follows from the lemma that $g = (9d - 33 \pm \sqrt{\Delta(d)})/6$. If $g = (9d - 33 - \sqrt{\Delta(d)})/6 \geqslant 0$ then $d \leqslant 13$, so $(d, g) \in \{(4, 0), (5, 2), (6, 3), (7, 5), (8, 6), (9, 6)\}$. On the other hand, if $g = (9d - 33 + \sqrt{\Delta(d)})/6$ then Halphen's bound yields $d \leqslant 20$ and hence $(d, g) \in \{(3, 1), (4, 1), (5, 2), (6, 4), (7, 5), (8, 7), (9, 10)\}$. If $(d, g) = (9, 6)$ then $\chi = -4$, so $X$ would be a ruled surface, and hence $K^2 = -31$ by the double-point formula. This contradicts the inequality $K^2 \leqslant 8\chi$. The rest of the cases are effective, and $X$ is well known in all of them. As $g$ is maximal (in the sense of [7]) except in the cases $(d, g) \in \{(4, 0), (5, 1), (6, 3), (8, 6)\}$, a simple description of $X$ and $\mathcal{I}_X$ follows by linkage. Moreover, if $(d, g) = (6, 3)$ then $X$ is linked to a cubic scroll by a complete intersection $(3, 3)$. If $(d, g) = (4, 0)$ then $h^1(\mathcal{I}_X(1)) = 1$, and hence $X$ is a projected Veronese surface by Severi's theorem [19]. Finally, in the cases $(d, g) \in \{(5, 1), (8, 6)\}$ one can easily describe $X$ as a surface linked to a Veronese surface by a complete intersection $(3, 3)$ and $(3, 4)$, respectively. □

**Remark 2.** Surfaces cut out by cubic hypersurfaces do not have any 4-secant line. Let us describe the family of 4-secant lines in the cases in which $X \subset \mathbb{P}^4$ is not cut out by cubic hypersurfaces, namely $(d, g) \in \{(8, 7), (8, 6)\}$:

(i) If $X$ is linked to a plane $X'$ by a c.i. $(3, 3)$, then it has a resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2} \to \mathcal{I}_X(4) \to 0.$$

In this case, $X$ is a minimal elliptic surface over $\mathbb{P}^1$ with Kodaira dimension $\kappa = 1$ (see [16] or [8]). It has a unique plane quartic curve $P \subset X'$, and it is fibered by the pencil $|H - P|$ of elliptic quartic curves.

(ii) If $X$ is linked to a Veronese surface by a c.i. $(3, 4)$ then it has a resolution:

$$0 \to T_{\mathbb{P}^4}(-2) \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(1) \to \mathcal{I}_X(4) \to 0.$$

In this case $\sigma : X \to \mathbb{P}^2$ is the blowing-up along 16 points $\{x_1, \ldots, x_4, y_1, \ldots, y_{12}\}$ lying on a quartic of $\mathbb{P}^2$ and embedded in $\mathbb{P}^4$ by the linear system $|\sigma^*(6L - \sum 2x_i - \sum y_j)|$ (see [16] or [8]). It has five plane quartic curves, namely $\sigma^*(4L - \sum x_i - \sum y_j)$ and $\sigma^*(5L - x_i - \sum_{k \neq i} 2x_k - \sum y_j)$, and it is ruled by five pencils of rational quartic curves, namely $|\sigma^*(2L - \sum x_i)|$ and $|\sigma^*(L - x_i)|$.

**Remark 3.** As expected, one can check that the Cayley–Le Barz formula (see [5] and [12]):

$$\frac{1}{12}(d-2)(d-3)^2(d-4) - \frac{1}{2}g\big(d^2 - 7d + 13 - g\big)$$

for the number, if finite, of 4-secant lines of $C \subset \mathbb{P}^3$ gives 1 in the case (i), where $(d, g) = (8, 7)$, and 5 in the case (ii), where $(d, g) = (8, 6)$.

**Remark 4.** If the family of 4-secant lines of a smooth surface $X \subset \mathbb{P}^4$ is at most 1-dimensional, then $C$ does not have any 4-secant line, so the Cayley–Le Barz formula and Halphen's bound yield

$$(d, g) \in \big\{(2, 0), (3, 0), (3, 1), (4, 0), (4, 1), (5, 1), (5, 2), (6, 3), (6, 4), (7, 5), (9, 10)\big\}$$

and hence $X$ is cut out by cubic hypersurfaces (cf. [4]).

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