C. R. Acad. Sci. Paris, Ser. I 351 (2013) 565–568

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Topology

On the hit problem for the polynomial algebra ☆

Sur le hit problem pour l'algèbre polynomiale

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A R T I C L E   I N F O

Article history:
Received 4 June 2013
Accepted after revision 17 July 2013
Available online 13 August 2013

Presented by the Editorial Board

Dedicated to Professor Huỳnh Mùi on the occasion of his seventieth birthday

A B S T R A C T

We study the *hit problem*, set up by F. Peterson, of finding a minimal set of generators for the polynomial algebra $P_k := \mathbb{F}_2[x_1, x_2, \dots, x_k]$ as a module over the mod-2 Steenrod algebra, $\mathcal{A}$. In this Note, we study a minimal set of generators for $\mathcal{A}$-module P_k in some so-called generic degrees and apply these results to explicitly determine the hit problem for $k = 4$.

R É S U M É

Nous étudions le problème suivant soulevé par F. Peterson : déterminer un système minimal de générateurs comme module sur l'algèbre de Steenrod pour l'algèbre polynomiale $P_k := \mathbb{F}_2[x_1, x_2, \dots, x_k]$, problème appelé *hit problem* en anglais. Dans ce but, nous étudions un ensemble minimal de générateurs pour le $\mathcal{A}$-module P_k dans certains degrés dits génériques. En appliquant ces résultats, nous déterminons explicitement le *hit problem* pour $k = 4$.

Let V_k be an elementary Abelian 2-group of rank k. Denote by BV_k the classifying space of V_k. It may be thought of as the product of k copies of the real projective space $\mathbb{RP}^\infty$. Then:

$$P_k := H^*(BV_k) \cong \mathbb{F}_2[x_1, x_2, \dots, x_k],$$

a polynomial algebra on k generators $x_1, x_2, \dots, x_k$, each of degree 1. Here the cohomology is taken with coefficients in the prime field $\mathbb{F}_2$ of two elements.

Being the cohomology of a space, P_k is a module over the mod 2 Steenrod algebra $\mathcal{A}$. The action of $\mathcal{A}$ on P_k is explicitly given by the formula:

$$Sq^i(x_j) = \begin{cases} x_j, & i = 0, \\ x_j^2, & i = 1, \\ 0, & \text{otherwise}, \end{cases}$$

and subject to the Cartan formula:

☆ The work was supported in part by a grant of NAFOSTED.
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1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2013.07.016

$$Sq^n(fg) = \sum_{i=0}^{n} Sq^i(f)Sq^{n-i}(g),$$

for $f, g \in P_k$ (see Steenrod and Epstein [12]).

A polynomial f in P_k is called *hit* if it can be written as a finite sum $f = \sum_{i>0} Sq^i(f_i)$ for some polynomials f_i. That means f belongs to $\mathcal{A}^+P_k$, where $\mathcal{A}^+$ denotes the augmentation ideal in $\mathcal{A}$. We are interested in the *hit problem*, set up by F. Peterson, of finding a minimal set of generators for the polynomial algebra P_k as a module over the Steenrod algebra. In other words, we want to find a basis of the $\mathbb{F}_2$-vector space $QP_k := P_k/\mathcal{A}^+.P_k = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$.

Let $GL_k = GL_k(\mathbb{F}_2)$ be the general linear group over the field $\mathbb{F}_2$. This group acts naturally on P_k by matrix substitution. Since the two actions of $\mathcal{A}$ and GL_k upon P_k commute with each other, there is an action of GL_k on QP_k. The subspace of degree n homogeneous polynomials $(P_k)_n$ and its quotient $(QP_k)_n$ are GL_k-subspaces of the spaces P_k and QP_k respectively.

The hit problem was first studied by Peterson [7], Wood [16], Singer [10,11], and Priddy [8], who showed its relationship to several classical problems respectively in the cobordism theory, modular representation theory, Adams spectral sequence for the stable homotopy of spheres, and stable homotopy type of classifying spaces of finite groups. The vector space QP_k was explicitly calculated by Peterson [7] for $k = 1, 2$, by Kameko [3] for $k = 3$. The case $k = 4$ has been treated by Kameko [4] and by us [13].

Several aspects of the hit problem were then investigated by many authors (e.g. Boardman, Bruner, Hưng, Carlisle, Wood, Crabb, Hubbuck, Peterson, Kameko, Nam, Singer, Walker and others).

The μ-function is one of the numerical functions that have much been used in the context of the hit problem. For a positive integer n, by $\mu(n)$ one means the smallest number r for which it is possible to write $n = \sum_{1\leqslant i\leqslant r}(2^{d_i} - 1)$, where $d_i > 0$. A routine computation shows that $\mu(n) = s$ if and only if there exist integers $d_1 > d_2 > \cdots > d_{s-1} \geqslant d_s > 0$ such that:

$$n = f(d_1, d_2, \ldots, d_s) := 2^{d_1} + 2^{d_2} + \cdots + 2^{d_{s-1}} + 2^{d_s} - s. \tag{1}$$

From this, it implies $n - s$ is even and $\mu(\frac{n-s}{2}) \leqslant s = \mu(n)$.

Peterson [7] made the following conjecture, which was subsequently proved by Wood [16].

Theorem 1. *(See Wood [16].) If $\mu(n) > k$, then $(QP_k)_n = 0$.*

One of the main tools in the study of the hit problem is the dual of the Kameko squaring $Sq^0_* : (QP_k)^{GL_k} \to (QP_k)^{GL_k}$. This homomorphism is induced by the following GL_k-homomorphism $\widetilde{Sq}^0_* : QP_k \to QP_k$. The latter is given by the $\mathbb{F}_2$-linear map, also denoted by $\widetilde{Sq}^0_* : P_k \to P_k$, given by:

$$\widetilde{Sq}^0_*(x) = \begin{cases} y, & \text{if } x = x_1x_2\ldots x_ky^2, \\ 0, & \text{otherwise}, \end{cases}$$

for any monomial $x \in P_k$. Note that $\widetilde{Sq}^0_*$ is not an $\mathcal{A}$-homomorphism. However,

$$\widetilde{Sq}^0_* Sq^{2t} = Sq^t \widetilde{Sq}^0_*,$$

for any nonnegative integer t.

Observe obviously that the homomorphism $\widetilde{Sq}^0_*$ is surjective on P_k and therefore on QP_k. So, one gets:

$$\dim(QP_k)_{2m+k} = \dim \operatorname{Ker}\big(\widetilde{Sq}^0_*\big)^k_m + \dim(QP_k)_m,$$

for any positive integer m. Here $(\widetilde{Sq}^0_*)^k_m : (QP_k)_{2m+k} \to (QP_k)_m$ denotes the squaring $\widetilde{Sq}^0_*$ in degree $2m + k$.

Theorem 2. *(See Kameko [3].) Let m be a positive integer. If $\mu(2m + k) = k$, then $(\widetilde{Sq}^0_*)^k_m : (QP_k)_{2m+k} \to (QP_k)_m$ is an isomorphism of GL_k-modules.*

Theorems 1 and 2 reduce the hit problem to the case of the degrees n with $\mu(n) = s < k$.

The hit problem in the case of degree n of the form (1) with $s = k - 1$, $d_{i-1} - d_i > 1$ for $2 \leqslant i < k$ and $d_{k-1} > 1$ was studied by Crabb and Hubbuck [2], Nam [5,6], and Repka and Selick [9].

In this Note, we explicitly determine the hit problem for the case $k = 4$. First, we study the hit problem for the cases of degree n of the form (1) for either $s = k - 1$ or $s = k - 2$. The following theorem gives an inductive formula for the dimension of $(QP_k)_n$ in this case.

Theorem 3. *Let $n = f(d_1, d_2, \ldots, d_{k-1})$ with d_i positive integers such that $d_1 > d_2 > \cdots > d_{k-2} \geqslant d_{k-1}$, and let $m = f(d_1 - d_{k-1}, d_2 - d_{k-1}, \ldots, d_{k-2} - d_{k-1})$. If $d_{k-1} \geqslant k - 1 \geqslant 1$, then:*

$$\dim(QP_k)_n = \big(2^k - 1\big)\dim(QP_{k-1})_m.$$

For $d_{k-1} \geqslant k$, the theorem follows from the results in Nam [5] and the present author [15]. However, for $d_{k-1} = k-1$, the theorem is new.

Based on Theorem 3, we explicitly compute QP_4.

Theorem 4. *Let n be an arbitrary positive integer with $\mu(n) < 4$. The dimension of the $\mathbb{F}_2$-vector space $(QP_4)_n$ is given by the following table*:

n	$s=1$	$s=2$	$s=3$	$s=4$	$s\geqslant 5$
$2^{s+1}-3$	4	15	35	45	45
$2^{s+1}-2$	6	24	50	70	80
$2^{s+1}-1$	14	35	75	89	85
$2^{s+2}+2^{s+1}-3$	46	94	105	105	105
$2^{s+3}+2^{s+1}-3$	87	135	150	150	150
$2^{s+4}+2^{s+1}-3$	136	180	195	195	195
$2^{s+t+1}+2^{s+1}-3,\ t\geqslant 4$	150	195	210	210	210
$2^{s+1}+2^{s}-2$	21	70	116	164	175
$2^{s+2}+2^{s}-2$	55	126	192	240	255
$2^{s+3}+2^{s}-2$	73	165	241	285	300
$2^{s+4}+2^{s}-2$	95	179	255	300	315
$2^{s+5}+2^{s}-2$	115	175	255	300	315
$2^{s+t}+2^{s}-2,\ t\geqslant 6$	125	175	255	300	315
$2^{s+2}+2^{s+1}+2^{s}-3$	64	120	120	120	120
$2^{s+3}+2^{s+2}+2^{s}-3$	155	210	210	210	210
$2^{s+t+1}+2^{s+t}+2^{s}-3, t\geqslant 3$	140	210	210	210	210
$2^{s+3}+2^{s+1}+2^{s}-3$	140	225	225	225	225
$2^{s+u+1}+2^{s+1}+2^{s}-3,\ u\geqslant 3$	120	210	210	210	210
$2^{s+u+2}+2^{s+2}+2^{s}-3,\ u\geqslant 2$	225	315	315	315	315
$2^{s+t+u}+2^{s+t}+2^{s}-3,\ u\geqslant 2, t\geqslant 3$	210	315	315	315	315

The space QP_4 was also computed in [4] by using computer calculation. However, the manuscript was unpublished at the time of the writing.

Carlisle and Wood showed in [1] that the dimension of the vector space $(QP_k)_m$ is uniformly bounded by a number depending on k. In 1990, Kameko made the following conjecture in his Johns Hopkins University PhD thesis [3].

Conjecture 5. *(See Kameko [3].) For every nonnegative integer m,*

$$\dim(QP_k)_m \leqslant \prod_{1\leqslant i\leqslant k}(2^i-1).$$

The conjecture was shown by Kameko himself for $k \leqslant 3$ in [3]. From Theorem 4, we see that the conjecture is also true for $k=4$.

By induction on k, using Theorem 3, we obtain the following.

Corollary 6. *Let $n = f(d_1, d_2, \dots, d_{k-1})$ with d_i positive integers. If $d_1 - d_2 \geqslant 2$, $d_{i-1} - d_i \geqslant i-1$, $3 \leqslant i \leqslant k-1$, $d_{k-1} \geqslant k-1$, then*:

$$\dim(QP_k)_n = \prod_{1\leqslant i\leqslant k}(2^i-1).$$

For the case $d_{i-1} - d_i \geqslant i$, $2 \leqslant i \leqslant k-1$, and $d_{k-1} \geqslant k$, this result is due to Nam [5]. This corollary also shows that Kameko's conjecture is true for the degree n as given in the corollary.

By induction on k, using Theorems 3, 4 and the fact that the dual of the Kameko squaring is an epimorphism, one gets the following.

Corollary 7. *Let $n = f(\lambda_1, \lambda_2, \dots, \lambda_{k-2})$ with λ_i positive integers and let $\lambda_{k-1} = 1$, $n_r = f(\lambda_1 - \lambda_{r-1}, \lambda_2 - \lambda_{r-1}, \dots, \lambda_{r-2} - \lambda_{r-1}) - 1$ with $r = 5, 6, \dots, k$. If $\lambda_1 - \lambda_2 \geqslant 4$, $\lambda_{i-2} - \lambda_{i-1} \geqslant i$, for $4 \leqslant i \leqslant k$ and $k \geqslant 5$, then*:

$$\dim(QP_k)_n = \prod_{1\leqslant i\leqslant k}(2^i-1) + \sum_{5\leqslant r\leqslant k}\Bigg(\prod_{r+1\leqslant i\leqslant k}(2^i-1)\Bigg)\dim\operatorname{Ker}\big(\widetilde{Sq}^0_*\big)^r_{n_r}.$$

Here, by convention, $\prod_{r+1\leqslant i\leqslant k}(2^i-1) = 1$ for $r=k$.

This corollary has been proved in [15] for the case $\lambda_{i-2} - \lambda_{i-1} > i+1$ with $3 \leqslant i \leqslant k$.

Obviously $2n_r + r = f(e_1, e_2, \dots, e_{r-2})$ where $e_i = \lambda_i - \lambda_{r-1} + 1$ for $1 \leqslant i \leqslant r-2$. So, in degree $2n_r + r$ of P_r, there exists a so-called spike $x = x_1^{2^{e_1}-1}x_2^{2^{e_2}-1}\dots x_{r-2}^{2^{e_{r-2}}-1}$, i.e. a monomial whose exponents are all of the form $2^e - 1$ for some e.

Since the class $[x]$ in $(QP_k)_{2n_r+r}$ represented by the spike x is nonzero and $(\widetilde{Sq}^0_*)^r_{n_r}([x]) = 0$, we have $(\mathrm{Ker}\,\widetilde{Sq}^0_*)^r_{n_r} \neq 0$, for any $5 \leqslant r \leqslant k$. Therefore, by Corollary 7, Kameko's conjecture is not true in degree $n = 2n_k + k$ for any $k \geqslant 5$, where $n_k = f(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{k-2} - 1) - 1$.

The first formulation of this Note was given in a 240-page preprint in 2007 [13], which was then publicized to a remarkable number of colleagues. One year later, we found the negative answer to Kameko's conjecture on the hit problem [14,15]. Being led by the insight of this new study, we have remarkably reduced the length of the paper.

The proofs of the results of this Note will be published in detail elsewhere.

Acknowledgements

It is a pleasure for me to express my hearty gratitude to Prof. Nguyễn H.V. Hưng for his valuable suggestions and constant encouragement. My thanks also go to all colleagues at the Department of Mathematics, Quy Nhơn University for many helpful conversations.

I would like to thank the referee for several helpful comments and suggestions, which have led to improvement of the paper's exposition.

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