



Algebraic Geometry

Mixed Hodge structures and Weierstrass  $\sigma$ -function*Structures de Hodge mixtes et fonction  $\sigma$  de Weierstrass*Grzegorz Banaszak<sup>a,1</sup>, Jan Milewski<sup>b</sup><sup>a</sup> Department of Mathematics and Computer Science, Adam Mickiewicz University, Poznań 61-614, Poland<sup>b</sup> Institute of Mathematics, Poznań University of Technology, ul. Piotrowo 3A, 60-965 Poznań, Poland

## ARTICLE INFO

## Article history:

Received 3 November 2012

Accepted 18 July 2013

Available online 20 August 2013

Presented by the Editorial Board

## ABSTRACT

A  $\sigma$ -operator on a complexification  $V_{\mathbb{C}}$  of an  $\mathbb{R}$ -vector space  $V_{\mathbb{R}}$  is an operator  $A \in \text{End}_{\mathbb{C}}(V_{\mathbb{C}})$  such that  $\sigma(A) = 0$ , where  $\sigma(z)$  denotes the Weierstrass  $\sigma$ -function. In this paper, we define the notion of strongly pseudo-real  $\sigma$ -operator and prove that there is a one-to-one correspondence between real mixed Hodge structures and strongly pseudo-real  $\sigma$ -operators.

© 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## R É S U M É

Un  $\sigma$ -opérateur sur la complexification  $V_{\mathbb{C}}$  d'un espace vectoriel réel  $V_{\mathbb{R}}$  est un opérateur  $A \in \text{End}_{\mathbb{C}}(V_{\mathbb{C}})$  tel que  $\sigma(A) = 0$ , où  $\sigma(z)$  est la fonction  $\sigma$  de Weierstrass. Dans cet article, nous introduisons la notion de  $\sigma$ -opérateur fortement pseudo-réel et démontrons qu'il y a une correspondance biunivoque entre les structures de Hodge mixtes réelles et les  $\sigma$ -opérateurs fortement pseudo-réels.

© 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## 1. Introduction

Let  $\mathbb{S} := R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$  and let  $V_{\mathbb{R}}$  be a finite-dimensional real vector space. By a real Hodge structure (HS) on  $V_{\mathbb{R}}$ , we understand a finite direct sum of real pure Hodge structures with given weights, cf. [1]. We can consider a Hodge structure as a real algebraic group representation  $\rho: \mathbb{S} \rightarrow GL(V_{\mathbb{R}})$  (see, e.g., [1,5,6] for the definition of HS). Let  $\mathcal{L}\rho$  be the Lie algebra representation of  $\rho$ . The following operator:

$$S := S(\rho) := \mathcal{L}\rho(1+i) \in \text{End}(V_{\mathbb{R}}),$$

introduced in [1], will be called the Hodge–Lie operator of the real HS given by  $\rho$ .

Let  $\sigma(z)$  be the Weierstrass sigma function for the lattice  $\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ , with  $\omega_1 = 1 - i$  and  $\omega_2 = 1 + i$ :

$$\sigma(z) := z \prod_{(p,q) \neq (0,0)} \left(1 - \frac{z}{\lambda_{p,q}}\right) \exp\left[\frac{z}{\lambda_{p,q}} + \frac{1}{2}\left(\frac{z}{\lambda_{p,q}}\right)^2\right]$$

where

$$\lambda_{p,q} := p\omega_1 + q\omega_2.$$

E-mail addresses: [banaszak@amu.edu.pl](mailto:banaszak@amu.edu.pl) (G. Banaszak), [jsmilew@wp.pl](mailto:jsmilew@wp.pl) (J. Milewski).

<sup>1</sup> Partially supported by the NCN (National Center of Science for Poland) NN201 607440.

**Definition 1.1.** Let  $V_K$  ( $K = \mathbb{R}$  resp.  $K = \mathbb{C}$ ) be a finite-dimensional vector space over  $K$ . An endomorphism  $A \in \text{End}(V_K)$  is called a (real resp. complex)  $\sigma$ -operator if  $\sigma(A) = 0$ .

Observe that if  $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$  then  $A$  is a  $\sigma$ -operator on  $V_{\mathbb{C}}$  if and only if  $\bar{A}$  is a  $\sigma$ -operator. In [1] we obtained the following result:

**Theorem 1.2.**

- (i) *The Hodge–Lie operator  $S$  of any HS is a real  $\sigma$ -operator.*
- (ii) *There is one-to-one correspondence between HS and the real  $\sigma$ -operators. This equivalence is given by assigning to an HS its Hodge–Lie operator:*

$$\rho \longleftrightarrow S(\rho).$$

Christopher Deninger asked us very interesting question about whether our approach to real HS via the Weierstrass  $\sigma$  function can be extended to the mixed Hodge structures (MHS). This paper is an affirmative answer to C. Deninger’s question (see Definition 1.5 and Theorem 1.6 below). For the definition of MHS, see, e.g., [2–4] (cf. Definition 2.1 below). In the whole of this paper, an MHS means a real MHS.

Let us introduce basic definitions and state the main results of this paper. The proofs of these results are given in Section 3. For every  $\lambda \in \Lambda$ , let us define:

$$\sigma_{\lambda}(z) := \frac{\sigma(z)}{z - \lambda}. \tag{1}$$

We will write  $\sigma_{m,n}(z) := \sigma_{\lambda_{m,n}}(z)$  for all  $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ .

**Definition 1.3.** Let  $V_{\mathbb{C}} := V_{\mathbb{R}} \otimes \mathbb{C}$ . A  $\sigma$ -operator  $A$  on  $V_{\mathbb{C}}$  is called a weakly pseudo-real if:

$$\sigma_{r,s}(A)(\bar{A} - A)\sigma_{p,q}(A) = 0 \tag{2}$$

for all  $r + s \geq p + q$ .

**Theorem 1.4.** *Every weakly pseudo-real  $\sigma$ -operator  $A$  determines an MHS.*

A natural question is whether there is a one-to-one correspondence between weakly pseudo-real  $\sigma$ -operators and real MHS on  $V_{\mathbb{R}}$ . In general the answer is no. However, if we strengthen the condition (2), then we can strengthen Theorem 1.4.

**Definition 1.5.** Let  $V_{\mathbb{C}} := V_{\mathbb{R}} \otimes \mathbb{C}$ . A  $\sigma$ -operator  $A$  on  $V_{\mathbb{C}}$  is called a strongly pseudo-real if:

$$\sigma_{r,s}(A)(\bar{A} - A)\sigma_{p,q}(A) = 0 \tag{3}$$

for all  $r \geq p$  or  $s \geq q$ .

**Theorem 1.6.** *There is a one-to-one correspondence between MHS and strongly pseudo-real  $\sigma$ -operators.*

Observe that  $A$  is a weakly pseudo-real (resp. strongly pseudo-real)  $\sigma$ -operator if and only if  $\bar{A}$  is a weakly pseudo-real (resp. strongly pseudo-real)  $\sigma$ -operator.

## 2. Deligne splitting of a mixed Hodge structure

**Definition 2.1.** A real MHS on  $V_{\mathbb{R}}$  consists of two filtrations, a finite increasing filtration on  $V_{\mathbb{R}}$ , the weight filtration  $W_{\bullet}^{\mathbb{R}}$  and a finite decreasing filtration  $F^{\bullet}$  on  $V_{\mathbb{C}} := V_{\mathbb{R}} \otimes \mathbb{C}$ , the Hodge filtration, which induce a pure real HS of the weight  $n$  on each graded piece:

$$\text{Gr}_n^W(V_{\mathbb{R}}) = W_n^{\mathbb{R}}/W_{n-1}^{\mathbb{R}}. \tag{4}$$

**Theorem 2.2** (Deligne splitting of MHS). *For any MHS, there exists exactly one decomposition:*

$$V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q} \tag{5}$$

such that

$$W_n = \bigoplus_{p+q \leq n} I^{p,q}, \quad F^p = \bigoplus_{\substack{k \geq p \\ q \in \mathbb{Z}}} I^{k,q}, \tag{6}$$

$$I^{p,q} = \overline{I^{q,p}} \pmod{D_{p-1,q-1}}, \tag{7}$$

where

$$W_n := W_n^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}, \quad D_{r,s} := \bigoplus_{\substack{k \leq r \\ l \leq s}} I^{k,l}. \tag{8}$$

This decomposition can be expressed via weight and Hodge filtrations in the following way:

$$I^{p,q} = V_{p+q}^p \cap (\overline{V_{p+q}^q} + \overline{U_{p+q-2}^{q-1}}), \tag{9}$$

where

$$V_n^p := F^p \cap W_n, \quad U_n^m := \sum_{j \geq 0} V_{n-j}^{m-j}. \tag{10}$$

For the proof of Theorem 2.2, see [3] (cf. [2], pp. 471–472). Observe that  $\overline{V_n^q} = \overline{F^q} \cap W_n$  because  $\overline{W_n} = W_n$ . Moreover:

$$D_{p,q} = \overline{D_{q,p}}, \tag{11}$$

$$W_n = \sum_{p+q=n} D_{p,q}. \tag{12}$$

### 3. Mixed Hodge structures via pseudo-real $\sigma$ -operators

Let  $A$  be a  $\sigma$ -operator on  $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$ . We get the following decomposition into eigenspaces:

$$V_{\mathbb{C}} := \bigoplus_{p,q} I_A^{p,q}, \quad I_A^{p,q} := \{x \in V : Ax = \lambda_{p,q}x\}. \tag{13}$$

Certainly

$$I_A^{p,q} = \overline{I_A^{q,p}} \tag{14}$$

because  $\lambda_{q,p} = \overline{\lambda_{p,q}}$ . Define the weight and Hodge filtrations and bifiltration of this  $\sigma$ -operator in the following way:

$$W_n^A := \bigoplus_{p+q \leq n} I_A^{p,q}, \quad F_A^p := \bigoplus_{\substack{k \geq p \\ q \in \mathbb{Z}}} I_A^{k,q}, \quad D_{p,q}^A := \bigoplus_{\substack{k \leq p \\ l \leq q}} I_A^{k,l}. \tag{15}$$

**Definition 3.1.** Two  $\sigma$ -operators  $A_1$  and  $A_2$  are called weakly equivalent if they determine the same weight filtration and they induce the same homomorphism on the grading of the weight filtration (hence their difference is a homomorphism of weight filtration of degree  $-1$ ).

**Lemma 3.2.** Let  $V_{\mathbb{C}}$  be a complexification of the real vector space  $V_{\mathbb{R}}$ . A  $\sigma$ -operator  $A \in \text{End}_{\mathbb{C}}(V_{\mathbb{C}})$  is weakly pseudo-real if and only if operators  $A$  and  $\overline{A}$  are weakly equivalent.

**Proof.** Consider the decomposition (13). The lemma is a consequence of the following equality:

$$\sigma_{p,q}(A) = \sigma_{p,q}(\lambda_{p,q})P_{p,q},$$

where  $P_{p,q}$  denotes the projection operator

$$P_{p,q} : \bigoplus_{m,n} I_A^{m,n} \rightarrow I_A^{p,q}$$

onto the direct summand  $I_A^{p,q}$ .  $\square$

**Proof of Theorem 1.4.** By Lemma 3.2, the  $\sigma$ -operators  $A$  and  $\bar{A}$  are weakly equivalent, hence they determine the same weight filtration  $W_\bullet = \overline{W_\bullet}$ , where  $W_\bullet := W_\bullet^A$  and  $\overline{W_\bullet} = W_\bullet^{\bar{A}}$ . Hence the weight filtration  $W_\bullet$  is a complexification of certain increasing filtration  $W_\bullet^{\mathbb{R}}$  on  $V_{\mathbb{R}}$ . Induced quotient operators  $[A]_n, [\bar{A}]_n$  on  $W_n/W_{n-1}$  are equal, hence are real  $\sigma$ -operators (a real  $\sigma$ -operator on  $V_{\mathbb{C}}$  is a  $\sigma$ -operator which is a complexification of an operator on  $V_{\mathbb{R}}$ ). All eigenvalues of the operator  $[A]_n$  have real part equal  $n$ , hence this operator gives a pure Hodge structure of weight  $n$ . The Hodge filtration of the  $\sigma$ -operator  $[A]_n$  on  $W_n/W_{n-1}$  is induced by the Hodge filtration of  $A$ . Hence the weight and Hodge filtrations of the  $\sigma$ -operator  $A$  induce a mixed Hodge structure.  $\square$

The double increasing, finite filtration  $D_{\bullet,\bullet}^A$  given in (15) leads to the following definition:

**Definition 3.3.** Two  $\sigma$ -operators  $A_1$  and  $A_2$  are called strongly equivalent if they determine the same double filtrations  $D_{\bullet,\bullet}^{A_1} = D_{\bullet,\bullet}^{A_2}$  and their difference is a homomorphism of this bifiltration of bidegree  $(-1, -1)$ .

**Lemma 3.4.** A  $\sigma$ -operator  $A$  is strongly pseudo-real if and only if operators  $A$  and  $\bar{A}$  are strongly equivalent.

**Proof.** The proof is very similar to the proof of Lemma 3.2 concerning the weakly pseudo-real  $\sigma$ -operator.  $\square$

**Proof of Theorem 1.6.** Every mixed Hodge structure  $(W_\bullet, F^\bullet)$  determines uniquely the canonical Deligne decomposition (5)–(8). Hence  $\sigma$ -operator  $A$ :

$$A := \bigoplus_{p,q} \lambda_{p,q} \text{id}_{I_{p,q}}$$

is also uniquely determined by this mixed Hodge structure and is strongly pseudo-real. Indeed:

$$\bar{A} = \bigoplus_{p,q} \lambda_{p,q} \text{id}_{I_{q,p}},$$

because  $\bar{\lambda}_{q,p} = \lambda_{p,q}$ . Recall that  $D_{q,p} = \bar{D}_{p,q}$ . Let  $x \in D_{p,q}$ . For  $r \leq p, s \leq q, k < r, l < s$ , there exist vectors  $\tilde{x}_{r,s} \in I^{s,r}, x_{r,s} \in I^{r,s}, u_{r,s;k,l} \in I^{k,l}$  such that:

$$x = \sum_{\substack{r \leq p \\ s \leq q}} \tilde{x}_{r,s}, \quad \tilde{x}_{r,s} = x_{r,s} + \sum_{\substack{k < r \\ l < s}} u_{r,s;k,l}.$$

Hence:

$$(\bar{A} - A)x = \sum_{r,s,k,l} (\lambda_{r,s} - \lambda_{k,l}) u_{r,s;k,l} \in D_{p-1,q-1},$$

where the sum  $\sum_{r,s,k,l}$  is given for indices  $r, s, k, l$  in the range  $r \leq p, s \leq q, k < r, l < s$ . By Lemma 3.4, the  $\sigma$ -operator  $A$  is strongly pseudo-real.

Conversely, every  $\sigma$ -operator  $A$  determines a unique decomposition of  $V_{\mathbb{C}}$  into eigen-subspaces (see (13) and (15)). When the  $\sigma$ -operator  $A$  is strongly pseudo-real, then, by Lemma 3.4, the equalities (13) and (15) give the Deligne canonical decomposition of the mixed Hodge structure determined by  $A$ . Indeed, in this case, for  $x \in I_A^{p,q}$ , we have  $\bar{A}x - Ax =: u \in D_{p-1,q-1}^A$ . Hence:

$$\bar{A}x = \lambda_{p,q}x + u. \tag{16}$$

It remains to check that:

$$x + u' \in \bar{I}_A^{q,p}, \tag{17}$$

for some  $u' \in D_{p-1,q-1}^A$ . Since  $A$  is strongly pseudo-real, we get  $D_{\bullet,\bullet}^A = D_{\bullet,\bullet}^{\bar{A}}$ . Observe that the operator  $\lambda_{p,q} \text{id} - \bar{A}$  with domain and target restricted to the invariant subspace  $D_{p-1,q-1}^A$  is an automorphism because  $\lambda_{p,q}$  does not belong to the spectrum of this restriction of  $A$ . Hence there exists unique  $u' \in D_{p-1,q-1}^A$  such that  $\lambda_{p,q}u' - \bar{A}u' = u$ . By (16) this vector  $u'$  satisfies the equality (17) because of (14). Observe that the following correspondences described in this proof:

MHS  $\longmapsto$  strongly pseudo-real  $\sigma$ -operator  
 strongly pseudo-real  $\sigma$ -operator  $\longmapsto$  MHS

are inverse one of the other.  $\square$

**Remark 3.5.** In [1] and in this paper we used the lattice  $\mathbb{Z}(1+i) + \mathbb{Z}(1-i)$  to determine the HS and MHS via the Weierstrass sigma function corresponding to this lattice. Take  $\omega = a + bi \in \mathbb{C}$  with  $ab \neq 0$ . Then  $\Lambda_\omega = \mathbb{Z}\omega + \mathbb{Z}\bar{\omega}$  is a lattice in  $\mathbb{C}$ . For this case one can also define the Hodge–Lie operator as follows:

$$S_\omega := S_\omega(\rho) := \mathcal{L}\rho(\omega) \in \text{End}(V_{\mathbb{R}}).$$

One observes that  $S_\omega$  is a  $\sigma_\omega$ -operator for the Weierstrass sigma function  $\sigma_\omega$  corresponding to  $\Lambda_\omega = \mathbb{Z}\omega + \mathbb{Z}\bar{\omega}$ . In the similar fashion as in [1] and in this paper we obtain one-to-one correspondence between real  $\sigma_\omega$ -operators and HS and one-to-one correspondence between strongly pseudo-real  $\sigma_\omega$ -operators and MHS.

## References

- [1] G. Banaszak, J. Milewski, Hodge structures and Weierstrass sigma-function, C. R. Acad. Sci. Paris, Ser. I 350 (2012) 777–780.
- [2] E. Cattani, A. Kaplan, W. Schmid, Degeneration of Hodge structures, Ann. Math. (2) 123 (3) (1986) 457–535.
- [3] P. Deligne, Structures de Hodge mixtes réelles. The appendix to “On the  $SL(2)$ -orbits in Hodge theory” by E. Cattani and A. Kaplan, IHES pre-pub./M/82/58 (1982).
- [4] P. Deligne, Structures de Hodge mixtes réelles, Proc. Symp. Pure Math. 55 (1994) 509–514.
- [5] B. Gordon, A survey of the Hodge conjecture for Abelian varieties, in: J. Lewis (Ed.), A Survey of the Hodge Conjecture, 1999, pp. 297–356 (Appendix B).
- [6] C. Peters, J. Steenbrink, Mixed Hodge Structures, Ergeb. Math. Grenzgeb., vol. 52, Springer, 2008.