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FE heterogeneous multiscale method for long-time wave propagation

Méthode d'éléments finis multi-échelles pour l'équation des ondes dans des milieux hétérogènes sur des temps longs

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ABSTRACT

A new finite element heterogeneous multiscale method (FE-HMM) is proposed for the numerical solution of the wave equation over long times in a rapidly varying medium. Our FE-HMM captures long-time dispersive effects of the true solution at a cost similar to that of a standard numerical homogenization scheme which, however, only captures the short-time macroscale behavior of the wave field.

RÉSUMÉ

Dans cet article, nous proposons une nouvelle méthode d'éléments finis multi-échelles pour la solution de l'équation des ondes dans des milieux hétérogènes sur des temps longs. Cette méthode numérique est capable d'approcher le comportement effectif de la solution sur des temps longs, avec un coût identique à celui d'une méthode d'homogénéisation numérique standard, qui ne peut capturer le comportement effectif de la solution que sur des temps courts.

Version française abrégée

Dans cet article, nous proposons une méthode d'éléments finis multi-échelles pour la résolution de l'équation des ondes dans des milieux hétérogènes (1) sur des temps longs. La résolution numérique efficace de ce type de problème est importante pour de nombreuses applications (imagerie médicale, inversion sismique, comportement élastique d'un matériau composite, etc.). L'homogénéisation de l'équation (1) est classique [8] et consiste à trouver une équation effective, où le terme $\nabla \cdot (a^\varepsilon \nabla u^\varepsilon)$ de (1) est remplacé par $\nabla \cdot (a^0 \nabla u^0)$. Nous rappelons que le tenseur homogénéisé a^0 est obtenu, dans le cas (localement) périodique, par une moyenne appropriée de la solution de problèmes microscopiques en chaque point x du domaine Ω. Une méthode d'homogénéisation numérique pour (1) a été proposée dans [4] avec un coût indépendant de ε.

Fixons maintenant $T_0 > 0$ et considérons la solution u^ε de (1) sur des temps longs, $T = T_0/\varepsilon^2$. On peut alors montrer que sur des intervalles $[0, T]$, la solution u^ε peut-être approchée (dans une norme L^∞) avec une erreur $\mathcal{O}(\varepsilon)$ par la résolution

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d'une équation effective dispersive (3) [13]. Auparavant, une autre équation effective (2) avait été proposée et obtenue avec des techniques de développement formel [15]. Nous remarquons que la solution numérique par la méthode proposée dans [4] n'est pas capable d'approcher la solution effective sur des temps longs. Des méthodes numériques pour calculer la solution effective de (2) ont été proposées dans [9] (basées sur le calcul des coefficients effectifs de (2)) et dans [11] (basées sur une méthode de différences finies multi-échelles de type [6]). Le désavantage de la méthode proposée dans [11] est qu'elle nécessite des domaines microscopiques qui augmentent en taille quand $\varepsilon \to 0$, un couplage micro–macro d'ordre élevé, et des corrections appropriées des conditions initiales. De plus, comme la solution de la méthode proposée dans [11] approche la solution d'une équation mal posée, des techniques de régularisation doivent être utilisées. La nouvelle méthode que nous proposons dans cet article est basée sur une méthode d'éléments finis multi-échelles de type [4]. Grâce à l'introduction d'un produit scalaire L^2 modifié, cette nouvelle méthode permet d'approcher la solution dispersive de (3) sur des temps longs avec un coût identique à celui de la méthode proposée dans [4], et bien inférieur à celui de la méthode proposée dans [11].

1. Introduction

The numerical solution of the wave equation:

$$\partial_{tt} u^{\varepsilon} - \nabla \cdot \left(a^{\varepsilon} \nabla u^{\varepsilon}\right) = F \quad \text{in } \Omega \times]0, T[, \tag{1}$$

with a rapidly varying coefficient $a^{\varepsilon}(x)$ is of fundamental importance for a wide range of applications (e.g., medical imaging, full-waveform seismic inversion, dynamic response of composite material). Here $\Omega \subset \mathbb{R}^d$, $u^{\varepsilon}(x,0) = f(x)$, $\partial_t u^{\varepsilon}(x,0) = g(x)$ in Ω, whereas the tensor $a^{\varepsilon}(x) \in (L^{\infty}(\Omega))^{d\times d}$ is symmetric and uniformly elliptic. Further ε represents a small scale in the medium, much smaller than any scale of interest such as the wavelength, which characterizes the multiscale nature of the problem.

As standard numerical methods require grid resolution of the medium down to its finest scales throughout Ω, they typically lead to prohibitively large problem size. In contrast, homogenization theory [8] provides the analytical framework for deriving a properly averaged (homogenized) field $a^0(x)$ that captures at the macroscopic scale the essential effects of the highly oscillatory velocity field $a^{\varepsilon}(x)$ as $\varepsilon \to 0$. Then, the homogenized solution, u^0, also satisfies (1) with a^{ε} replaced by a^0. Since explicit formulas for a^0 are only available in a few situations (e.g., periodic or random stationary fields), numerical multiscale methods that overcome these limitations are needed. In [14,16], for instance, effective coarse scale models are computed from the fully resolved wave equation (1) throughout Ω; hence, the initial set-up cost for the coarse (upscaled) model increases as $\varepsilon \to 0$. In contrast, the finite element heterogeneous multiscale method (FE-HMM) (see e.g., [2,7]) computes an effective wave equation at the macroscale from elliptic micro problems on sampling domains of size $\delta = \mathcal{O}(\varepsilon)$; hence, the computational cost is independent of ε. At finite time and for a locally periodic medium, it also yields optimal convergence to the limit u^0 from classical homogenization theory with decreasing mesh size.

For limited time, the propagation of waves in a highly oscillatory medium is well-described by the classical homogenized wave equation. With increasing time, however, the true solution, u^{ε}, deviates from the classical homogenization limit, u^0, as dispersive effects develop – see Fig. 1 below. To capture those dispersive effects over long times $T = \mathcal{O}(1/\varepsilon^2)$, Santosa and Symes [15] devised in one space dimension and for periodic a^{ε} the effective Boussinesq-type equation:

$$\partial_{tt} u^{\text{eff}} - a^0 \partial_{xx} u^{\text{eff}} - \varepsilon^2 b^0 \partial_{xxxx} u^{\text{eff}} = F. \tag{2}$$

It was rederived by formal asymptotic expansion in [9]. Recently Engquist, Holst and Runborg [11] proposed a finite difference (FD) HMM approach, which overcomes the limitations from precomputed effective coefficients. The FD-HMM is able to capture those long-time dispersive effects, but it also requires increasingly larger space–time sampling domains as $\varepsilon \to 0$, together with high-order macro–micro coupling and correction to the initial data. Moreover, since the FD-HMM solution converges with decreasing mesh size to (2), which is ill-posed, regularization is also needed.

In [13], Lamacz rigorously proved that u^{ε} can be approximated with error $\mathcal{O}(\varepsilon)$ (in an L^{∞} norm) on the time interval $T = \mathcal{O}(1/\varepsilon^2)$ by the solution u^{eff} of a well-posed one-dimensional limit equation:

$$\partial_{tt} u^{\text{eff}} - a^0 \partial_{xx} u^{\text{eff}} - \varepsilon^2 \frac{b^0}{a^0} \partial_{tt} \partial_{xx} u^{\text{eff}} = F. \tag{3}$$

Note that (3) coincides with (2) if time derivatives are formally replaced by space derivatives in the third term. By using Bloch-wave techniques, that analysis was recently extended to higher dimensions [10]. The weak formulation of (3) suggests that an effective correction at the macro-scale is also needed in the L^2 inner product term that involves $\partial_{tt} u^{\text{eff}}$.

The finite-element heterogeneous multiscale method introduced in this paper yields on the fly a numerical approximation for a Boussinesq-type equation, such as (3), yet at a cost identical to that of a standard FE-HMM [4]. In particular, thanks to a modified L^2 scalar product, our new FE-HMM is even able to capture dispersive effects of the true solution at later times.

2. FE heterogeneous multiscale method

Since the FE-HMM proposed in [4] converges to u^0, it also fails to capture the long-time dispersive effects in the true solution, u^ε, as illustrated in Fig. 1 below. To incorporate those dispersive effects, we shall modify the L^2 inner product, thereby mimicking the weak formulation of (3). This new FE-HMM method, denoted by FE-HMM-L (for long time), relies on the same micro problems as the original FE-HMM from [4]; hence, the computational cost is identical.

Before we present the FE-HMM-L method, we start with some notations. Let $\{\mathcal{T}_H\}$ be a family of macro triangulations[1] (e.g., simplicial elements) of the computational domain Ω ($H \gg \varepsilon$ is allowed) and $S^\ell(\Omega, \mathcal{T}_H)$ an FE space of piecewise polynomials of maximum degree ℓ. Next, within each macro element $K \in \mathcal{T}_H$, we choose an appropriate quadrature formula (QF) with nodes $x_{K,j}$ and positive weights $\omega_{K,j}$, which satisfies standard ellipticity and approximation conditions [2]. At each quadrature point $x_{K,j}$, we also consider sampling domains $K_\delta = x_{K,j} + \delta I$, where $I = (-1/2, 1/2)^d$ and $\delta \geqslant \varepsilon$. On the sampling domain K_δ, we choose a micro FE space $S^q(K_\delta, \mathcal{T}_h)$ with micro triangulation $\mathcal{T}_h$, and denote by $v_{H,\mathrm{lin}}$ the linearization of $v_H \in S^\ell(\Omega, \mathcal{T}_H)$ about $x_{K,j}$. The FE-HMM-L method now reads as follows: find $u_H : [0, T] \to S^\ell(\Omega, \mathcal{T}_H)$ such that:

$$\begin{cases} \partial_{tt}\big((u_H, v_H) + (u_H, v_H)_M\big) + B_H(u_H, v_H) = (F, v_H), \quad \forall v_H \in S^\ell(\Omega, \mathcal{T}_H), \\ u_H(0) = f_H, \qquad \partial_t u_H(0) = g_H \quad \text{in } \Omega, \end{cases} \tag{4}$$

where the initial data f_H and g_H are suitable approximations of f and g in $S^\ell(\Omega, \mathcal{T}_H)$. In (4), the FE-HMM bilinear form is defined as:

$$B_H(v_H, w_H) = \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{K,j}}{|K_\delta|} \int_{K_\delta} a^\varepsilon(x) \nabla v_h(x) \cdot \nabla w_h(x)\, \mathrm{d}x, \tag{5}$$

and the long-time correction by:

$$(v_H, w_H)_M = \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{K,j}}{|K_\delta|} \int_{K_\delta} \big(v_h(x) - v_{H,\mathrm{lin}}(x)\big)\big(w_h(x) - w_{H,\mathrm{lin}}(x)\big)\, \mathrm{d}x. \tag{6}$$

In (5), (6), the micro solutions v_h (resp. w_h) are given by: find v_h such that $(v_h - v_{H,\mathrm{lin}}) \in S^q(K_\delta, \mathcal{T}_h)$ and:

$$\int_{K_\delta} a^\varepsilon(x) \nabla v_h(x) \cdot \nabla z_h(x)\, \mathrm{d}x = 0, \quad \forall z_h \in S^q(K_\delta, \mathcal{T}_h). \tag{7}$$

Thanks to the $(v_H, w_H)_M$ correction in the L^2 inner product, the FE-HMM-L scheme is now able to recover the dispersive behavior at later times – see Fig. 1. In practice, the remaining two standard L^2 scalar products $(\cdot,\cdot)$ in (4) are also evaluated via numerical quadrature, which, however, does not need to coincide with that used to compute either B_H or the required L^2-correction.

In the following proposition, we show that the L^2-correction indeed has order $\mathcal{O}(\varepsilon^2)$ and that $(\cdot,\cdot) + (\cdot,\cdot)_M$ is a true inner product. Hence, the FE-HMM-L method in (4) is well-defined for all $\varepsilon, H, h > 0$.

Proposition 2.1. *The L^2-correction in (4) is positive semi-definite*:

$$(v_H, v_H)_M \geqslant 0, \quad \forall v_H \in S^\ell(\Omega, \mathcal{T}_H).$$

Furthermore the correction is of order $\mathcal{O}(\varepsilon^2)$:

$$\big|(v_H, w_H)_M\big| \leqslant C\varepsilon^2 \|\nabla v_H\|_{L^2} \|\nabla w_H\|_{L^2}.$$

Proof. The positivity immediately follows from the positivity of the quadrature weights $\omega_{K,j}$. To show the second statement, we rewrite the solution of (7) as:

$$v_h = v_{H,\mathrm{lin}} + \delta \sum_{i=1}^{d} \hat{\psi}_h^i \left(\frac{x - x_{K,j}}{\delta} \right) \partial_{x_i} v_{H,\mathrm{lin}}, \tag{8}$$

where $\hat{\psi}_h^i \in S^q(Y)$ solves the cell-problem

$$\int_Y a_{x_{K,j}}(y) \nabla \hat{\psi}_h^i(y) \cdot \nabla \hat{z}_h\, \mathrm{d}y = -\int_Y a_{x_{K,j}}(y) e_i \cdot \nabla \hat{z}_h\, \mathrm{d}y, \quad \forall \hat{z}_h \in S^q(Y),$$

[1] Clearly, the method also applies to hexahedral elements with obvious changes.

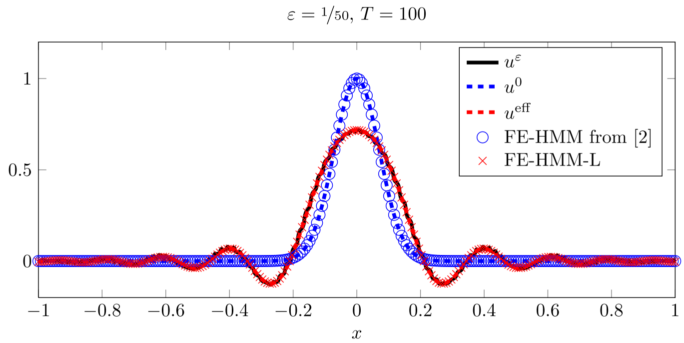

Fig. 1. The solution u^ε of (1), the solution u^0 from classical homogenization, the solution u^{eff} of (3), the numerical FE-HMM solution from [4], and the new FE-HMM-L solution of (4) are shown at time $T = 100$ for $\varepsilon = 1/50$. Note that u^ε, u^{eff} and the FE-HMM-L solution coincide.

with $a_{x_{K,j}}(y) = a(x_{K,j} + \delta y)$ and $y \in Y = (-1/2, 1/2)^d$. By using (8) in (6) we thus obtain:

$$(v_H, w_H)_M = \sum_{K\in\mathcal{T}_H}\sum_{j=1}^{J}\omega_{K,j}\sum_{r,s=1}^{d}\delta^2\partial_{x_r}v_H(x_{K,j})\partial_{x_s}w_H(x_{K,j})\int_Y \hat{\psi}_h^r(y)\hat{\psi}_h^s(y)\,\mathrm{d}y.$$

As $a_{x_{K,j}}$ is uniformly elliptic and bounded, the H^1-norm of $\hat{\psi}_h^i$ is also bounded and we finally have:

$$\begin{aligned}(v_H, w_H)_M &\leqslant C\varepsilon^2\frac{\delta^2}{\varepsilon^2}\left(\sum_{K\in\mathcal{T}_H}\sum_{j=1}^{J}\omega_{K,j}\left|\nabla v_H(x_{K,j})\right|^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_H}\sum_{j=1}^{J}\omega_{K,j}\left|\nabla w_H(x_{K,j})\right|^2\right)^{\frac{1}{2}}\\ &\leqslant C\varepsilon^2\|\nabla v_H\|_{L^2}\|\nabla w_H\|_{L^2}\end{aligned}$$

since δ is proportional to ε. $\square$

By using Proposition 2.1 we can prove the following convergence result, see [5] for more details.

Theorem 2.1. *Let u^0, u_H be the homogenized and the FE-HMM-L solutions, respectively. Under appropriate assumptions on the quadrature formula and the regularity of u^0, we have*:

$$\left\|\partial_t\left(u^0 - u_H\right)\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|u^0 - u_H\right\|_{L^\infty(0,T;H^1(\Omega))} \leqslant C\left(H^\ell + e_{\text{HMM}} + \varepsilon^2\right),$$

and

$$\left\|u^0 - u_H\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\left(H^{\ell+1} + e_{\text{HMM}} + \varepsilon^2\right),$$

for all $H \leqslant H_0$ and $\varepsilon \leqslant \varepsilon_0$, where e_{HMM} is the standard FE-HMM error, which can be decomposed into the modeling error e_{MOD} and the micro error e_{MIC}.

We note that the micro error e_{MIC} can be bounded by $C(h/\varepsilon)^{2q}$ provided sufficient regularity of the oscillating tensor, while the modeling error $e_{\text{MOD}} = 0$ for locally periodic tensors with appropriate size of the sampling domains and periodic boundary conditions for the micro problem (see [1,3] for further details). Hence as ε, H, and $h \to 0$, the FE-HMM-L solution u_H also converges to u^0 for finite time. Yet for fixed values $h, H, \varepsilon > 0$, it also captures the dispersive effects that emerge at later times in u^ε, as demonstrated in the following numerical experiment.

3. Numerical experiment

We consider (1) in $\Omega = (-1, 1)$ with periodic boundary conditions, let $u(x, 0)$ be a Gaussian pulse with zero initial velocity and:

$$a^\varepsilon = \sqrt{2} + \sin\left(2\pi\frac{x}{\varepsilon}\right) \tag{9}$$

with $\varepsilon = 1/50$. In Fig. 1, the solution u^ε of (1), the solution u^0 from classical homogenization, the solution u^{eff} of (3), the numerical FE-HMM solution from [4], and the new FE-HMM-L solution of (4) are shown at time $T = 100$ for $\varepsilon = 1/50$. In particular, we observe the dispersive effects that appear in the (fully resolved numerical) solution u^ε at later times. Because of the periodicity of the medium, the homogenized tensor $a^0 = 1$; hence, the corresponding homogenized solution u^0 is non-dispersive and can be computed analytically. The solution u^{eff} is computed on a coarse mesh with $H = 2^{-8}$, where the coefficient $b^0 = 9.09632625 \cdot 10^{-3}$ in (3) is computed with MAPLE [12]. To minimize numerical dispersion, we use cubic finite elements at the macro and the micro scale, with mesh sizes $H = 2^{-8}$ and $h = \varepsilon/100$, respectively, but lower-order elements could also be used. Both FE-HMM schemes use standard second-order finite differences for time discretization. In contrast to the FE-HMM solution from [4], which approximates u^0, the solution from the new FE-HMM-L scheme coincides with u^ε and u^{eff} even at later times, thus exhibiting the correct dispersive behavior.

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