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Nonlinear Donati compatibility conditions for the nonlinear Kirchhoff–von Kármán–Love plate theory

Conditions de compatibilité non linéaires de Donati pour la théorie non linéaire des plaques de Kirchhoff–von Kármán–Love

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ABSTRACT

Let ω be a simply-connected domain in $\mathbb{R}^2$ and let $(E_{\alpha\beta})$ and $(F_{\alpha\beta})$ be two symmetric 2×2 matrix fields with components in $L^2(\omega)$. In this Note, we identify nonlinear compatibility conditions "of Donati type" that the components $E_{\alpha\beta}$ and $F_{\alpha\beta}$ must satisfy in order that there exists a vector field $(\eta_1, \eta_2, w) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ such that:

$$\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha w \partial_\beta w) = E_{\alpha\beta} \quad \text{and} \quad \partial_{\alpha\beta} w = F_{\alpha\beta} \quad \text{in } \omega.$$

The left-hand sides of these relations are the components of tensors found in the Kirchhoff–von Kármán–Love theory of nonlinearly elastic plates.

RÉSUMÉ

Soit ω un domaine simplement connexe de $\mathbb{R}^2$ et soient $(E_{\alpha\beta})$ et $(F_{\alpha\beta})$ deux champs de matrices 2×2 symétriques dont les composantes sont dans $L^2(\omega)$. Dans cette Note, on identifie et justifie des conditions non linéaires de compatibilité « de type Donati » que doivent satisfaire les composantes $E_{\alpha\beta}$ et $F_{\alpha\beta}$ afin qu'il existe un champ de vecteurs $(\eta_1, \eta_2, w) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ tel que :

$$\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha w \partial_\beta w) = E_{\alpha\beta} \quad \text{et} \quad \partial_{\alpha\beta} w = F_{\alpha\beta} \quad \text{dans } \omega.$$

Les membres de gauche de ces relations sont les composantes de tenseurs trouvés dans la théorie de Kirchhoff–von Kármán–Love des plaques non linéairement élastiques.

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1. Preliminaries

Greek indices vary in $\{1,2\}$ and the convention summation with respect to repeated indices is used. A *domain* in $\mathbb{R}^2$ is a bounded, open, and connected subset ω of $\mathbb{R}^2$ with a Lipschitz-continuous boundary $\partial\omega$, the set ω being locally on the same side of $\partial\omega$. Partial derivatives of the first, second, and third, order of functions of $y=(y_\alpha)\in\omega$ are denoted $\partial_\alpha := \partial/\partial y_\alpha$, $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$, and $\partial_{\alpha\beta\sigma} := \partial^3/\partial y_\alpha \partial y_\beta \partial y_\sigma$; the same notations are used for partial derivatives in the sense of distributions.

Vector fields and matrix fields, and spaces of vector fields, defined over ω are denoted by boldface letters. The usual Sobolev spaces over ω are denoted $H^m(\omega)$, $m\in\mathbb{Z}$, and $H_0^m(\omega)$, $m\geqslant 1$; the norm in $H^m(\omega)$, $m\in\mathbb{Z}$, is denoted $\|\cdot\|_{m,\omega}$; in particular then, $\|\cdot\|_{0,\omega}$ is the norm of $H^0(\omega)=L^2(\omega)$. The notation $\mathbb{L}^2(\omega)$ designates the space of all 2×2 *symmetric* matrix fields with components in $L^2(\omega)$. If $\boldsymbol{S}=(S_{\alpha\beta})$ is a 2×2 matrix field with smooth enough components defined over ω, its *divergence* $\mathbf{div}\,\boldsymbol{S}$ is the vector field defined by $(\mathbf{div}\,\boldsymbol{S})_\alpha = \partial_\beta S_{\alpha\beta}$.

If X and Y are two (real) vector spaces and A is a linear operator from X to Y,

$$\operatorname{Im} A := \{y\in Y;\ y=Ax \text{ for at least one } x\in X\} \quad \text{and} \quad \operatorname{Ker} A := \{x\in X;\ Ax=0\}.$$

The notation ${}_{X'}\langle\cdot,\cdot\rangle_X$ designates the duality between a normed vector space X and its dual X'.

In the classical *Kirchhoff–von Kármán–Love theory of nonlinearly elastic plates* (see, e.g., Chapters 4 and 5 in [2]), the unknown displacement field of the middle surface $\overline{\omega}$ of the plate minimizes an energy whose integrand contains a positive-definite quadratic function of the *change of metric* and *change of curvature tensors*, respectively defined by

$$E_{\alpha\beta} := \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha w \partial_\beta w) \quad \text{and} \quad F_{\alpha\beta} := \partial_{\alpha\beta} w, \tag{1}$$

for an arbitrary displacement field $(\eta_1,\eta_2,w)\in H_0^1(\omega)\times H_0^1(\omega)\times H_0^2(\omega)$ (we consider here plates that are clamped over their entire lateral face).

In the *intrinsic* approach to the same theory, the matrix fields $(E_{\alpha\beta})\in\mathbb{L}^2(\omega)$ and $(F_{\alpha\beta})\in\mathbb{L}^2(\omega)$ are considered as the *sole unknowns*. There thus arises the question as to whether there exist suitable *compatibility conditions* that the components $E_{\alpha\beta}$ and $F_{\alpha\beta}$ of these matrix fields should satisfy in order that there exists a vector field $(\eta_1,\eta_2,w)\in H_0^1(\omega)\times H_0^1(\omega)\times H_0^2(\omega)$ satisfying (1). As shown in [6], if the domain ω is simply-connected, the **nonlinear Saint-Venant compatibility conditions**:

$$\begin{aligned}
&\partial_{\sigma\tau}E_{\alpha\beta} + \partial_{\alpha\beta}E_{\sigma\tau} - \partial_{\alpha\sigma}E_{\beta\tau} - \partial_{\beta\tau}E_{\alpha\sigma} + F_{\alpha\beta}F_{\sigma\tau} - F_{\alpha\sigma}F_{\beta\tau} = 0 \quad \text{in } H^{-2}(\omega),\\
&\partial_\sigma F_{\alpha\beta} - \partial_\beta F_{\alpha\sigma} = 0 \quad \text{in } H^{-1}(\omega),
\end{aligned}$$

constitute one possible answer to this question. The objective of this Note is to give (cf. Theorem 4.2) a different answer to the same question, this time in the form of *variational equations*, which as such constitute examples of **nonlinear Donati compatibility conditions** (a general presentation of Saint-Venant and Donati compatibility conditions as they arise in three-dimensional linearized elasticity is found in Chapter 6 in [3]).

Complete proofs and an application to the intrinsic nonlinear plate theory will be found in [5].

2. An existence theorem for an Airy-function

The following result is a "weak" version (already used in [6]) of a classical result for smooth functions. Its proof is based on the two-dimensional version of the *weak Poincaré lemma* due to [4] and then given a substantially simpler proof in [10]; cf. also Theorem 6.17-4 in [3].

Theorem 2.1. *Let ω be a simply-connected domain in $\mathbb{R}^2$, and let there be given a matrix field $(F_{\alpha\beta})\in\mathbb{L}^2(\omega)$ that satisfies*

$$\partial_\sigma F_{\alpha\beta} - \partial_\beta F_{\alpha\sigma} = 0 \quad \textit{in } H^{-1}(\omega).$$

Then there exists a function $\varphi\in H^2(\omega)$, unique up to the addition of a polynomial of degree $\leqslant 1$, such that

$$\partial_{\alpha\beta}\varphi = F_{\alpha\beta} \quad \textit{in } L^2(\omega).$$

Theorem 2.1 can be immediately recast as an existence result of an ad hoc *Airy function* (denoted φ in the next theorem) under low regularity assumptions. As such, it complements Theorem 2 of [7], where the existence of an Airy function was established, again in the space $H^2(\omega)$, but for *non-simply-connected domains*, under the assumption that the tensor field noted $\boldsymbol{S}$ in the next theorem satisfies in addition the usual global equilibrium equations.

Theorem 2.2. *Let ω be a simply-connected domain in $\mathbb{R}^2$, and let there be given a matrix field $\boldsymbol{S}=(S_{\alpha\beta})\in\mathbb{L}^2(\omega)$ that satisfies*

$$\mathbf{div}\,\boldsymbol{S} = \mathbf{0} \quad \textit{in } \boldsymbol{H}^{-1}(\omega).$$

Then there exists a function $\varphi \in H^2(\omega)$, unique up to the addition of a polynomial of degree $\leqslant 1$, such that

$$\partial_{11}\varphi = S_{22}, \qquad \partial_{12}\varphi = -S_{12}, \qquad \partial_{22}\varphi = S_{11} \quad \textit{in } L^2(\omega). \tag{2}$$

A function φ satisfying the relations (2) is called an **Airy function associated with the matrix field $\boldsymbol{S}$**.

3. Linear Donati compatibility conditions for linearly elastic plates

For convenience, we consider separately the existence of the "horizontal" components η_α, and that of the "vertical" component w of the unknown vector field.

Theorem 3.1. *Let ω be a domain in $\mathbb{R}^2$ and let there be given a matrix field $(e_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ that satisfies*

$$\int_\omega e_{\alpha\beta} s_{\alpha\beta}\,\mathrm{d}y = 0 \quad \textit{for all } (s_{\alpha\beta}) \in \mathbb{L}^2(\omega) \textit{ such that } \partial_\beta s_{\alpha\beta} = 0 \textit{ in } H^{-1}(\omega). \tag{3}$$

Then there exists a vector field $(\eta_\alpha) \in H_0^1(\omega) \times H_0^1(\omega)$ such that

$$\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) = e_{\alpha\beta} \quad \textit{in } L^2(\omega),$$

and such a vector field (η_α) is uniquely determined.

Proof. See, e.g., [1], or [8] and [9]. □

Theorem 3.2. *Let ω be a domain in $\mathbb{R}^2$ and let there be given a matrix field $(F_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ that satisfies*

$$\int_\omega F_{\alpha\beta} T_{\alpha\beta}\,\mathrm{d}y = 0 \quad \textit{for all } (T_{\alpha\beta}) \in \mathbb{L}^2 \textit{ such that } \partial_{\alpha\beta} T_{\alpha\beta} = 0 \textit{ in } H^{-2}(\omega).$$

Then there exists a function $w \in H_0^2(\omega)$ such that

$$\partial_{\alpha\beta} w = F_{\alpha\beta} \quad \textit{in } L^2(\omega), \tag{4}$$

and this function is uniquely determined.

Sketch of proof. Let the continuous linear operator $\boldsymbol{H} : H_0^2(\omega) \to \mathbb{L}^2(\omega)$ be defined by

$$\boldsymbol{H} w = \begin{pmatrix} \partial_{11} w & \partial_{12} w \\ \partial_{21} w & \partial_{22} w \end{pmatrix} \in \mathbb{L}^2(\omega) \quad \text{for each } w \in H_0^2(\omega).$$

Then one shows that $\operatorname{Im} \boldsymbol{H}$ is a closed subspace of $\mathbb{L}^2(\omega)$, and that the dual operator of $\boldsymbol{H}$ is $\operatorname{div}\mathbf{div} : \mathbb{L}^2(\omega) \to H^{-2}(\omega)$. The conclusion then follows from Banach closed range theorem. □

Relations (3) and (4) constitute the **linear Donati compatibility conditions** *corresponding to the Kirchhoff–Love theory of linearly elastic plates*. Note that they hold regardless of whether the domain ω is simply-connected.

4. Nonlinear Donati compatibility conditions for nonlinearly elastic plates

The *Green's formula* (5) established in the next theorem is crucial to our subsequent analysis.

Theorem 4.1. *For all functions $w \in H_0^2(\omega)$ and $\varphi \in H^2(\omega)$,*

$$\int_\omega (\partial_{11} w \partial_{22} w - \partial_{12} w \partial_{12} w)\varphi\,\mathrm{d}y = \int_\omega \left\{ -\frac{1}{2}(\partial_1 w)^2 \partial_{22}\varphi - \frac{1}{2}(\partial_2 w)^2 \partial_{11}\varphi + \partial_1 w \partial_2 w \partial_{12}\varphi \right\} \mathrm{d}y. \tag{5}$$

Sketch of proof. Both sides of (5) being continuous functions of $(w, \varphi) \in H_0^2(\omega) \times H^2(\omega)$, it is enough to establish (5) for all $(w, \varphi) \in \mathcal{D}(\omega) \times H^2(\omega)$. To this end, one uses the integration by parts formulas in Sobolev spaces. □

The next theorem constitutes the main result of this Note.

Theorem 4.2. *Let ω be a simply-connected domain in $\mathbb{R}^2$. Given a matrix field $\boldsymbol{S} \in \mathbb{L}^2(\omega)$ that satisfies $\mathbf{div}\,\boldsymbol{S} = \mathbf{0}$ in $\boldsymbol{H}^{-1}(\omega)$, there exists a unique function $\varphi \in H^2(\omega)$ such that (cf. Theorem 2.2):*

$$\partial_{11}\varphi = S_{22}, \qquad \partial_{12}\varphi = -S_{12}, \qquad \partial_{22}\varphi = S_{11} \quad \text{in } L^2(\omega) \quad \text{and} \quad \int_\omega \varphi \,\mathrm{d}y = \int_\omega \partial_\alpha \varphi \,\mathrm{d}y = 0.$$

Let

$$\Phi : \left\{\boldsymbol{S} \in \mathbb{L}^2(\omega);\ \mathbf{div}\,\boldsymbol{S} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\omega)\right\} \to \left\{\psi \in H^2(\omega);\ \int_\omega \psi \,\mathrm{d}y = \int_\omega \partial_\alpha \psi \,\mathrm{d}y = 0\right\}$$

denote the mapping defined in this fashion, i.e., by $\Phi(\boldsymbol{S}) := \varphi$.

Let there be given two matrix fields $(E_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ and $\boldsymbol{F} = (F_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ that satisfy

$$\int_\omega F_{\alpha\beta} T_{\alpha\beta} \,\mathrm{d}y = 0 \quad \text{for all } \mathbf{T} = (T_{\alpha\beta}) \in \mathbb{L}^2(\omega) \text{ such that } \operatorname{div}\mathbf{div}\,\boldsymbol{T} = 0 \text{ in } H^{-2}(\omega), \tag{6}$$

$$\int_\omega \left\{E_{\alpha\beta} S_{\alpha\beta} + (\det \boldsymbol{F})\Phi(\boldsymbol{S})\right\} \mathrm{d}y = 0 \quad \text{for all } \mathbf{S} = (S_{\alpha\beta}) \in \mathbb{L}^2(\omega) \text{ such that } \mathbf{div}\,\boldsymbol{S} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\omega). \tag{7}$$

Then there exists a vector field $(\eta_1, \eta_2, w) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ such that

$$\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha w \partial_\beta w) = E_{\alpha\beta} \quad \text{in } L^2(\omega),$$
$$\partial_{\alpha\beta} w = F_{\alpha\beta} \quad \text{in } L^2(\omega),$$

and such a vector field (η_1, η_2, w) is uniquely determined.

Proof. Relation (6) shows that there exists a uniquely determined function $w \in H_0^2(\omega)$ such that (cf. Theorem 3.2)

$$F_{\alpha\beta} = \partial_{\alpha\beta} w \quad \text{in } L^2(\omega). \tag{8}$$

Given any matrix field $\boldsymbol{S} = (S_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ such that $\mathbf{div}\,\boldsymbol{S} = \mathbf{0}$ in $\boldsymbol{H}^{-1}(\omega)$, there exists a uniquely determined function $\varphi \in H^2(\omega)$ such that (cf. Theorem 2.2):

$$S_{11} = \partial_{22}\varphi, \qquad S_{12} = -\partial_{12}\varphi, \qquad S_{22} = \partial_{11}\varphi \quad \text{in } L^2(\omega).$$

Therefore, for any such matrix field $\boldsymbol{S}$,

$$\int_\omega E_{\alpha\beta} S_{\alpha\beta} \,\mathrm{d}y = \int_\omega \{E_{11}\partial_{22}\varphi + E_{22}\partial_{11}\varphi - 2E_{12}\partial_{12}\varphi\} \,\mathrm{d}y,$$

and (cf. (8))

$$\begin{aligned}\int_\omega (\det \boldsymbol{F})\Phi(\boldsymbol{S}) \,\mathrm{d}y &= \int_\omega \left(F_{11}F_{22} - (F_{12})^2\right)\varphi \,\mathrm{d}y \\ &= \int_\omega \left\{(\partial_{11} w \partial_{22} w - \partial_{12} w \partial_{12} w)\varphi \,\mathrm{d}y\right\}.\end{aligned}$$

The left-hand side of relation (6) can thus be rewritten as (cf. (5)):

$$\begin{aligned}&\int_\omega \left\{E_{\alpha\beta} S_{\alpha\beta} + (\det \boldsymbol{F})\boldsymbol{\Phi}(\boldsymbol{S})\right\} \mathrm{d}y \\ &\quad = \int_\omega \left\{\left(E_{11} - \frac{1}{2}(\partial_1 w)^2\right)\partial_{22}\varphi - 2\left(E_{12} - \frac{1}{2}\partial_1 w \partial_2 w\right)\partial_{12}\varphi + \left(E_{22} - \frac{1}{2}(\partial_2 w)^2\right)\partial_{11}\varphi\right\} \mathrm{d}y \\ &\quad = \int_\omega \left\{\left(E_{11} - \frac{1}{2}(\partial_1 w)^2\right)S_{11} + 2\left(E_{22} - \frac{1}{2}\partial_1 w \partial_2 w\right)S_{12} + \left(E_{22} - \frac{1}{2}(\partial_2 w)^2\right)S_{22}\right\} \mathrm{d}y.\end{aligned}$$

Since this last relation holds for all $\boldsymbol{S} = (S_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ such that $\mathbf{div}\,\boldsymbol{S} = \boldsymbol{0}$ in $\boldsymbol{H}^{-1}(\omega)$, there exists a uniquely determined vector field $(\eta_\alpha) \in H_0^1(\omega) \times H_0^1(\omega)$ such that (cf. Theorem 3.1):

$$E_{\alpha\beta} - \frac{1}{2}\partial_\alpha w \partial_\beta w = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) \quad \text{in } L^2(\omega).$$

This completes the proof. □

Relations (6) and (7) constitute the **nonlinear Donati compatibility conditions** *corresponding to the Kirchhoff–von Kármán–Love theory of nonlinearly elastic plates*. Note that, when properly extended, they can also cover the case where the domain ω is not simply-connected; cf. [5].

Finally, note that, as expected, *the linearization of the nonlinear Donati compatibility conditions* (7) *reduce to the linear ones* (cf. (3)).

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