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A smooth extension method

Une méthode de prolongement régulier

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A R T I C L E I N F O

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A B S T R A C T

In this note, we present a smooth extension method for the simulation of the motion of immersed rigid bodies. It is a method of the fictitious domain type, which uses Cartesian meshes and recovers the optimal order of the error by finding a smooth extension of the exact solution defined in the domain with holes. We first present the method with a Poisson problem and show next how it can be adapted to the case of immersed rigid bodies. Finally, the method is validated in both the scalar and the vector cases.

R É S U M É

Nous présentons dans cette note une méthode de prolongement régulier pour simuler le mouvement de particules rigides immergées dans un fluide incompressible. C'est une méthode de type domaine fictif sur maillage cartésien permettant de retrouver l'ordre optimal de l'erreur en espace, en trouvant un prolongement régulier de la solution exacte définie sur le domaine perforé. Nous présentons tout d'abord la méthode sur un problème scalaire, puis nous l'adaptons au cas des équations de Stokes incompressibles et des particules rigides. Elle est ensuite validée sur différents cas de test.

Version française abrégée

Nous présentons dans cette note une méthode de type domaine fictif, permettant de simuler le mouvement de particules rigides immergées dans un fluide incompressible, dont le comportement est régi par les équations de Stokes. Le champ de vitesse global (prolongé par le mouvement rigide dans les inclusions) n'étant pas régulier, la précision en espace est dégradée lorsque l'on utilise des méthodes classiques de pénalisation ou multiplicateurs de Lagrange sur maillage non conforme. Dans le contexte variationnel, la méthode de la frontière élargie [10,7], les méthodes basées sur une formulation à la Nitsche [5,6,8], ou encore les approches de type contrôle [2–4,11] permettent de récupérer cet ordre optimal. La méthode présentée ici est de type contrôle optimal et a pour but de retrouver l'ordre optimal de l'erreur sans modifier l'opérateur du laplacien discret sur le maillage cartésien, de façon à permettre l'utilisation de solveurs rapides. L'approche est basée sur la construction d'un prolongement régulier de la solution sur tout le domaine, par la recherche d'un second membre adapté.

Nous présentons ici le principe de la méthode dans le cas scalaire, et nous renvoyons à la section 3 pour le cas du problème de Stokes.

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Soient Ω un domaine de $\mathbb{R}^n$ et B une boule incluse dans Ω. On considère le problème jouet défini par les équations (1), avec f dans $L^2(\Omega \setminus \bar{B})$. L'idée de la méthode est de construire un prolongement régulier de u à Ω tout entier en trouvant un minimiseur de la fonctionnelle (2), où u_g est solution du problème scalaire (3) posé sur le domaine global Ω. Les fonctions $\chi_{\Omega\setminus\bar{B}}$ et χ_B désignent respectivement l'indicatrice de $\Omega \setminus \bar{B}$ et l'indicatrice de B.

Proposition 0.1. *Si $\Omega \setminus \bar{B}$ est lipschitzien, il existe une fonction g dans $L^2(B)$ telle que la restriction de u_g, solution de (3), à $\Omega \setminus \bar{B}$, soit égale à la solution u de (1). Cette fonction g n'est pas unique.*

Le gradient de J est donné dans la proposition qui suit :

Proposition 0.2. *Soit g dans $L^2(B)$ et u_g la solution du problème (3). On considère le problème de Poisson (4), où le second membre est une distribution de simple couche définie par :*

$$\langle u_g \delta_{\partial B}, v\rangle = \int_{\partial B} u_g v \quad \forall v \in H_0^1(\Omega).$$

Le gradient de J est alors :

$$\nabla J(g) = w_{g|B}.$$

Si on utilise un algorithme de type gradient conjugué pour minimiser la fonctionnelle J, l'algorithme pour trouver la solution u est le suivant :

1. Étant donnée une fonction g dans $L^2(B)$, trouver la solution u_g de (3) :
$$\begin{cases} -\Delta u_g = f\chi_{\Omega\setminus\bar{B}} + g\chi_B & \text{dans } \Omega, \\ u_g = 0 & \text{sur } \partial\Omega. \end{cases}$$
2. Calculer la solution w_g du problème (4) :
$$\begin{cases} -\Delta w_g = u_g \delta_{\partial B} & \text{dans } \Omega, \\ w_g = 0 & \text{sur } \partial\Omega. \end{cases}$$
3. Prendre la restriction de w_g à B pour obtenir le gradient de J et mettre à jour le contrôle g.

Cette méthode de prolongement régulier demande donc de résoudre deux problèmes à chaque itération de l'algorithme de minimisation. Elle permet l'utilisation d'un maillage cartésien et ne modifie pas les opérateurs de type laplacien qui interviennent dans les deux problèmes. Il est donc possible de résoudre ces problèmes de Poisson par des transformées de Fourier rapides, ou bien par des algorithmes de type multigrille géométrique. De plus, on peut vérifier que cette méthode récupère l'ordre optimal de l'erreur (voir partie 4).

1. Introduction

In this note, we present a method of the fictitious domain type to simulate the motion of immersed rigid bodies. The fluid obeys the incompressible Stokes equations. Methods like the classical penalty method or the Lagrange multipliers method are known to produce a solution with a non-optimal order of the error, in particular in the neighborhood of the immersed bodies. In order to recover this optimal order, other methods exist in the finite element framework, such as the Fat Boundary Method [10,7], method based on Nitsche's formulation [5,6,8] or control approach methods [2–4,11]. The method presented here is in the spirit of these control approach methods and aims at recovering the optimal error order while using Cartesian meshes to allow the use of fast solvers. The idea is to find a smooth extension of the exact solution to the whole domain by finding a suitable extension of the right-hand side in the inclusions. We first present this smooth extension method with a Poisson equation and show the differences in the case of the Stokes system.

2. A smooth extension method

Let Ω be an open subspace of $\mathbb{R}^n$ and B a sphere included in Ω. We consider the following toy problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \setminus \bar{B}, \\ u = 0 & \text{on } \partial(\Omega \setminus \bar{B}), \end{cases} \tag{1}$$

where f is in $L^2(\Omega \setminus \bar{B})$. The solution u is thus in the Sobolev space $H^2(\Omega \setminus \bar{B})$. The method described here enforces the Dirichlet condition on the boundary ∂B by finding a smooth extension of u to the whole domain Ω.

Let us consider the following functional defined in $L^2(B)$:

$$J(g) = \frac{1}{2}\int_{\partial B} |u_g|^2, \tag{2}$$

where u_g is the solution of the following Poisson problem:

$$\begin{cases} -\Delta u_g = f\chi_{\Omega\setminus\bar{B}} + g\chi_B & \text{in } \Omega, \\ u_g = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

The functions $\chi_{\Omega\setminus\bar{B}}$ and χ_B are respectively the indicator functions of $\Omega\setminus\bar{B}$ and B. The right-hand side of (3) is in $L^2(\Omega)$ so that the solution u_g is in $H^2(\Omega)$. The idea is to find a minimizer g of J so that u_g is a smooth extension of u to the whole domain Ω.

Proposition 2.1. *Suppose that $\Omega\setminus\bar{B}$ has a Lipschitz boundary, there exists a function g in $L^2(B)$ such that the restriction of u_g to $\Omega\setminus\bar{B}$ is the solution u of (1).*

Proof. The solution u is in $H^2(\Omega\setminus\bar{B})$. Applying Stein's theorem (see [1]), there exists a smooth extension $\tilde{u}$ of u to the whole domain Ω. Now, we choose:

$$g = -\Delta\tilde{u}_{|B},$$

and the solution u_g is thus equal to the extension $\tilde{u}$ of u. □

Remark 1. The control g is obviously not unique. Indeed, one can add any function in $H_0^2(B)$ to the extension $\tilde{u}_{|B}$ to get another smooth extension of u.

Because we want to find a function g such that:

$$g = \underset{\phi\in L^2(B)}{\arg\min}\, J(\phi) = \frac{1}{2}\int_{\partial B} |u_\phi|^2,$$

we are now interested in the gradient of the functional J.

Proposition 2.2. *Let g be in $L^2(B)$ and u_g be the solution of problem (3). We consider the following Poisson problem*:

$$\begin{cases} -\Delta w_g = u_g\delta_{\partial B} & \text{in } \Omega, \\ w_g = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where the right-hand side is a single-layer distribution defined by:

$$\langle u_g\delta_{\partial B}, v\rangle = \int_{\partial B} u_g v \quad \forall v \in H_0^1(\Omega).$$

The gradient of J is:

$$\nabla J(g) = w_{g|B}.$$

In order to use a conjugate gradient algorithm to find a minimizer of J, one needs to compute, at each iteration, the gradient of J. The algorithm to compute the gradient reads as follows:

1. Given a function g in $L^2(B)$, find the solution u_g of (3)

$$\begin{cases} -\Delta u_g = f\chi_{\Omega\setminus\bar{B}} + g\chi_B & \text{in } \Omega, \\ u_g = 0 & \text{on } \partial\Omega. \end{cases}$$

2. Find the solution w_g of the gradient problem (4):

$$\begin{cases} -\Delta w_g = u_g\delta_{\partial B} & \text{in } \Omega, \\ w_g = 0 & \text{on } \partial\Omega. \end{cases}$$

3. Take the restriction of w_g to B to get the gradient and update the control g.

Remark 2. At each iteration of the conjugate gradient algorithm, one needs to solve two Poisson problems. Nevertheless, this method is of the fictitious domain type and one can use Cartesian meshes to use fast solvers. Moreover, the operators in these two Poisson problems do not depend on the position of the sphere B; only the right-hand side does. Therefore, one can use solvers such as the *Fast Fourier Transform* or geometric multigrids or even compute an LU factorization at the beginning and use it throughout the simulation.

3. The smooth extension method for the Stokes problem

In this section, we present the fluid problem and the differences with the scalar toy problem (1). Let Ω be a domain in $\mathbb{R}^d$ and B_i a collection of balls included in Ω. The problem writes:

$$\begin{cases} -2\nu\nabla\cdot\left(\dfrac{\nabla\mathbf{u}+{}^t\nabla\mathbf{u}}{2}\right)+\nabla p=\mathbf{f} & \text{in } \Omega\setminus\bar{B},\\ \nabla\cdot\mathbf{u}=0 & \text{in } \Omega\setminus\bar{B},\\ \mathbf{u}=0 & \text{on } \partial\Omega,\\ \mathbf{u}=\mathbf{V}_i+\boldsymbol{\omega}_i\times(\mathbf{x}-\mathbf{x}_i) & \text{on } \partial B_i,\\ \displaystyle\int_{\partial B_i}\sigma(\mathbf{u})\cdot\mathbf{n}_i=\mathbf{F}_i, & \\ \displaystyle\int_{\partial B_i}(\mathbf{x}-\mathbf{x}_i)\times\sigma(\mathbf{u})\cdot\mathbf{n}_i=0, & \end{cases} \tag{5}$$

where B is the union of all the balls B_i and $\mathbf{f}$ is in $L^2(\Omega\setminus\bar{B})^d$. The unknowns of this problem are $\mathbf{u}$ the velocity of the fluid, p the pressure, V_i the velocity of the i-th rigid particle and $\boldsymbol{\omega}_i$ its angular velocity. The center of the i-th particle is denoted by $\mathbf{x}_i$ and $\mathbf{F}_i$ is a force applied on the i-th particle like the gravity in the case of a sedimentation simulation. The tensor $\sigma(\mathbf{u})$ is the stress tensor:

$$\sigma(\mathbf{u})=2\nu D(\mathbf{u})-pI_d=2\nu\left(\frac{\nabla\mathbf{u}+{}^t\nabla\mathbf{u}}{2}\right)-pI_d,$$

where I_d is the identity matrix. The first four equations of the system (5) are the incompressible Stokes equations and their boundary conditions. On the boundary of the particles, we use the no-slip condition, hence the expression given in the system. The last two equations come from the equilibrium of the force and the torque for the i-th particle.

Given a function $\mathbf{g}$ in $L^2(\Omega\setminus\bar{B})^d$, we consider the following Stokes problem in the whole domain Ω:

$$\begin{cases} -2\nu\nabla\cdot\left(\dfrac{\nabla\mathbf{u}_g+{}^t\nabla\mathbf{u}_g}{2}\right)+\nabla p_g=\mathbf{f}\chi_{\Omega\setminus\bar{B}}+\mathbf{g}\chi_B+\displaystyle\sum_i\frac{\mathbf{F}_i}{|B_i|}\chi_{B_i} & \text{in } \Omega,\\ \nabla\cdot\mathbf{u}_g=0 & \text{in } \Omega,\\ \mathbf{u}_g=0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

The smooth extension method applied to this problem consists in solving the following optimization problem:

Find $\mathbf{g}\in L^2(\Omega\setminus\bar{B})^d$ such that,

$$\mathbf{g}=\underset{\mathbf{v}\in L^2(\Omega\setminus\bar{B})^d}{\arg\min}\ J(\mathbf{v})=\sum_i\int_{\partial B_i}\big(\mathbf{u_v}-\mathcal{R}_i(\mathbf{u_v})\big)^2, \tag{7}$$

where $\mathcal{R}_i(\mathbf{u_v})$ is the rigid motion associated with $\mathbf{u_v}$:

$$\mathcal{R}_i(\mathbf{u_v})=\frac{1}{|\partial B_i|}\int_{\partial B_i}\mathbf{u_v}+\frac{d}{2R_i^2|\partial B_i|}\left(\int_{\partial B_i}(\mathbf{x}-\mathbf{x}_i)\times\mathbf{u_v}\right)\times(\mathbf{x}-\mathbf{x}_i), \tag{8}$$

where R_i is the radius of the i-th particle, and $|\partial B_i|$ is the length of the boundary of B_i.

Like in the scalar case, one can show that a minimizer exists. One difference with the scalar case is stated in the following proposition:

Proposition 3.1. *Let $\mathbf{g}$ be a minimizer of J. The solution $\mathbf{u}_g$ of (6) satisfies the last two equations of (5) if and only if $\mathbf{g}$ is in the following constrained space K:*

$$K=\left\{\mathbf{g}\in L^2(\Omega\setminus\bar{B})^d;\ \int_{B_i}\mathbf{g}=0\ \text{and}\ \int_{B_i}(\mathbf{x}-\mathbf{x}_i)\times\mathbf{g}=0\right\}.$$

The restriction of $\mathbf{u}_g$ to $\Omega\setminus\bar{B}$ is thus the solution $\mathbf{u}$ of (5).

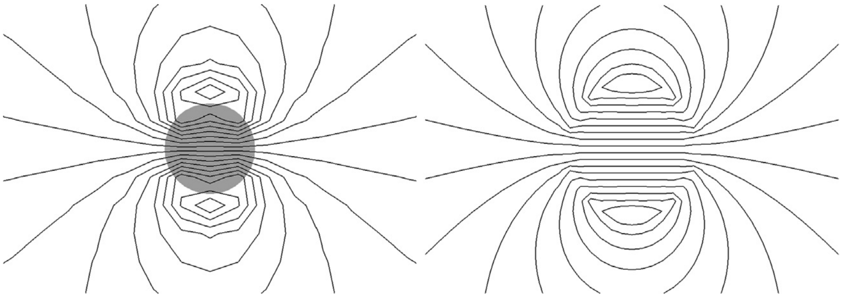

Fig. 1. Isovalues of the pressure for a coarse mesh (left) and a fine mesh (right).

Fig. 1. Isovaleurs de la pression pour un maillage grossier (à gauche) et un maillage fin (à droite).

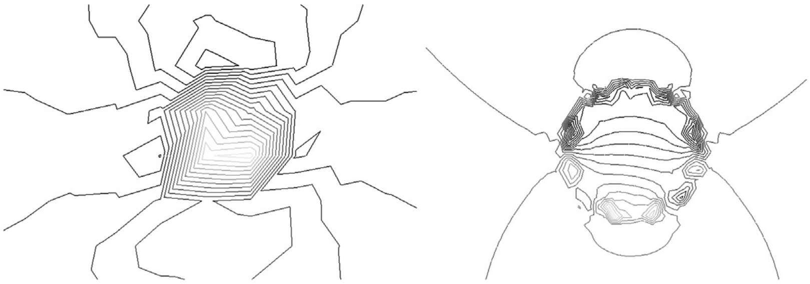

Fig. 2. Isovalues of the pressure with a penalty method.

Fig. 2. Isovaleurs de la pression avec une méthode de pénalisation.

4. Validation of the method

We present here the evolution of the H^1 error with respect to the mesh size h of the following problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus \bar{B}, \\ u = u_{\partial\Omega} & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{9}$$

where Ω is the unit square in two dimensions, B is the ball included in Ω and centered at $\mathbf{x}_c$ with radius R, and the function $u_{\partial\Omega}$ is defined by:

$$u_{\partial\Omega} = \log\left(\frac{|\mathbf{x} - \mathbf{x}_c|}{R}\right). \tag{10}$$

The exact solution of problem (9) is known and is the function defined by (10). Numerically, we find a minimizer of the functional (2) with a conjugate gradient algorithm. The error is computed on a mesh of size $h = 2^{-10}$ and the problem is solved using this smooth extension method with mesh sizes h from 2^{-5} to 2^{-9}. An order of 1.05 is recovered numerically in this range of mesh sizes.

In the case of the Stokes equation, we consider a single rigid particle sedimenting in the unit square. Fig. 1 (right) shows the pressure contour lines of the solution when the unit square is meshed with 2^7 points on each boundary. See also in Fig. 1 (left) the pressure contour lines of the solution, in the case where the unit square is meshed with only 33 points on each boundary, so that the disc diameter scales like 3 or 4 elements.

To illustrate the difference with methods that natively rely on a non-smooth extension of the velocity field, we plot in Fig. 2 the same isovalues, computed with a penalty method (see e.g. [9]).

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