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Differential Geometry

Curvature properties of anti-Kähler–Codazzi manifolds

Propriétés de courbure des variétés anti-Kähler–Codazzi

Arif Salimov [a], Sibel Turanli [b]

[a] *Ataturk University, Faculty of Science, Department of Mathematics, 25240, Erzurum, Turkey*
[b] *Erzurum Technical University, Faculty of Science, Department of Mathematics, Erzurum, Turkey*

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ABSTRACT

In this paper we shall consider a new class of integrable almost anti-Hermitian manifolds, which will be called anti-Kähler–Codazzi manifolds, and we will investigate their curvature properties.

RÉSUMÉ

Dans cet article, nous allons considérer une nouvelle classe de variétés intégrables presque anti-hermitiennes qui seront appelées variétés anti-Kähler–Codazzi, et nous allons étudier les propriétés de courbure de ces variétés.

1. Introduction

Let (M, J) be a $2n$-dimensional almost complex manifold, where J denotes its almost complex structure. A semi-Riemannian metric g of neutral signature (n, n) is an anti-Hermitian (Norden) metric if:

$$g(JX, Y) = g(X, JY)$$

for any $X, Y \in \aleph(M)$, where $\aleph(M)$ is the module of vector fields on M. An almost complex manifold (M, J) with an anti-Hermitian metric is referred to as an almost anti-Hermitian manifold. Structures of this kind have been also studied under the name: almost complex structures with pure (or B-) metric. An anti-Kähler (Kähler–Norden) manifold can be defined as a triple (M, g, J), which consists of a smooth manifold M endowed with an almost complex structure J and an anti-Hermitian metric g such that $\nabla J = 0$, where ∇ is the Levi-Civita connection of g. It is well known that the condition $\nabla J = 0$ is equivalent to C-holomorphicity (analyticity) of the anti-Hermitian metric g [1], i.e. $\Phi_J g = 0$, where Φ_J is the Tachibana operator [4]: $(\Phi_J g)(X, Y, Z) = (L_{JX} g - L_X G)(Y, Z)$, where $G(Y, Z) = (g \circ J)(Y, Z) = g(JY, Z)$ is the twin anti-Hermitian metric. It is a remarkable fact that (M, g, J) is anti-Kähler if and only if the twin anti-Hermitian structure (M, G, J) is anti-Kähler. This is of special significance for anti-Kähler metrics since in such case g and G share the same Levi-Civita connection. Since in dimension 2 an anti-Kähler manifold is flat, we assume in the sequel that $\dim M \geqslant 4$.

Let now (M, g, J) be an almost anti-Hermitian manifold and let $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ be the curvature operator of the Levi-Civita connection ∇ on M. Then the Ricci tensor S is defined as $S(X, Y) = trace\{Z \to R(Z, X)Y\}$. We note that for the case where (M, g, J) is anti-Kähler manifold these tensors have the following properties [1]:

E-mail addresses: asalimov@atauni.edu.tr (A. Salimov), sibelturanli@hotmail.com (S. Turanli).

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$$J\big(R(X,Y)Z\big)=R(JX,Y)Z=R(X,JY)Z=R(X,Y)JZ,\qquad S(JX,Y)=S(X,JY),$$

i.e. R and S are pure tensors with respect to the structure J (for more details about pure tensors, see [3]). Moreover, in such a manifold, R and S are C-holomorphic tensors.

2. Anti-Kähler–Codazzi manifolds

It is well known that the pair (J,g) of an almost Hermitian structure defines a fundamental 2-form Ω by $\Omega(X,Y)=g(JX,Y)$. If the skew-symmetric tensor Ω is a Killing–Yano tensor, i.e.

$$(\nabla_X\Omega)(Y,Z)+(\nabla_Y\Omega)(X,Z)=0 \tag{1}$$

or equivalently if the almost complex structure J satisfies $(\nabla_X J)Y+(\nabla_Y J)X=0$ for any $X,Y\in\aleph(M)$, then the manifold is called a nearly Kähler manifold (or K-space).

Let now (M,g,J) be an almost anti-Hermitian manifold. Then the pair (J,g) defines, as usual, the twin anti-Hermitian metric $G(Y,Z)=(g\circ J)(Y,Z)=g(JY,Z)$, but G is symmetric, rather than a 2-form Ω. Thus, the anti-Hermitian pair (J,g) does not give rise to a 2-form, and the Killing–Yano equation (1) has no immediate meaning. Therefore, we can replace the Killing–Yano equation by Codazzi equation:

$$(\nabla_X G)(Y,Z)-(\nabla_Y G)(X,Z)=0. \tag{2}$$

Eq. (2) is equivalent to:

$$(\nabla_X J)Y-(\nabla_Y J)X=0. \tag{3}$$

If the almost complex structure of almost anti-Hermitian manifold satisfies (3), then the triple (M,J,g) is called an anti-Kähler–Codazzi manifold (or *AKC*-space).

Remark 1. Let the tensor G (i.e. the twin anti-Hermitian metric) be a Killing symmetric tensor, i.e. $\underset{X,Y,Z}{\sigma}(\nabla_X G)(Y,Z)=0$, where σ is the cyclic sum with respect to X,Y and Z. This is the class of the quasi-Kähler manifold with anti-Hermitian (Norden) metric [2].

Theorem 2.1. *Anti-Kähler–Codazzi manifolds have integrable almost anti-Hermitian structures.*

Proof. Using $\nabla_X Y-\nabla_Y X=[X,Y]$, $(\nabla_X J)(JY)=-J(\nabla_X J)Y$ for every almost anti-Hermitian manifold and (3), we have:

$$\begin{aligned}N_J(X,Y)&=[JX,JY]-J[X,JY]-J[JX,Y]-[X,Y]\\&=\nabla_{JX}JY-\nabla_{JY}JX-J(\nabla_X JY-\nabla_{JY}X)-J(\nabla_{JX}Y-\nabla_Y JX)+J^2(\nabla_X Y-\nabla_Y X)\\&=-J\big((\nabla_X J)Y-(\nabla_Y J)X\big)+(\nabla_{JX}J)Y-(\nabla_{JY}J)X\\&=-J\big((\nabla_{JY}J)JX-(\nabla_{JX}J)JY\big)+J\big((\nabla_Y J)X-(\nabla_X J)Y\big)=0,\end{aligned}$$

i.e. the Nijenhuis tensor N_J vanishes. Conversely, from property $N_J=0$ not conclude (3). The proof of the theorem is complete. □

3. Curvature properties

Let the triple (M,g,J) be an anti-Kähler–Codazzi manifold. Since ∇_X commutes with every contraction (trace) of a tensor field and *trace* $J=0$, we have from (3):

$$\begin{aligned}q&=trace\big\{V\to(\nabla_V J)X-(\nabla_X J)V\big\}\\&=trace\big\{V\to(\nabla_V J)X\big\}-\nabla_X\, trace\, J\\&=trace\big\{V\to(\nabla_V J)X\big\}=0.\end{aligned}$$

Let $x^1,\dots,x^{2n}$ be a local coordinate system in M. By setting $V=\frac{\partial}{\partial x^i}$ and $X=\frac{\partial}{\partial x^j}$, $i,j=1,\dots,2n$, in this equation, we have $q_j=\nabla_i J_j^i=0$.

Applying the Ricci identity to the tensor field J, we find:

$$\nabla_k\nabla_j J_i^h-\nabla_j\nabla_k J_i^h=R_{kjt}{}^{h}J_i^t-R_{kji}^t J_t^h,$$

where R_{kji}^h are components of curvature tensor R. After contraction with respect to k and h in this equation, by virtue of $q_j=0$, we have:

$$\begin{aligned}\nabla_h \nabla_j J_i^h &= S_{jt} J_i^t - R_{hji}^t J_t^h = S_{jt} J_i^t - R_{hjil} g^{lt} J_t^h \\ &= S_{jt} J_i^t - R_{hjil} G^{lh} = S_{jt} J_i^t - H_{ji},\end{aligned} \tag{4}$$

where S_{jt} are the components of the Ricci tensor S, G^{lh} are the contravariant components of twin anti-Hermitian metric G and $H_{ji} = R_{hjil} G^{lh}$. Since $G^{lh} = G^{hl}$, $R_{(hj)il} = 0$, $R_{hj(il)} = 0$, from $H_{ji} = R_{hjil} G^{lh}$ we have:

$$H_{ji} = \frac{1}{2}(R_{hjil} + R_{ljih}) G^{lh} = \frac{1}{2}(R_{hjil} + R_{ihlj}) G^{lh}$$

or

$$H_{ji} - H_{ij} = \frac{1}{2}(R_{hjil} - R_{jhli} + R_{ihlj} - R_{hijl}) G^{lh} = 0,$$

i.e. H is a symmetric tensor field. Then, by virtue of $H_{[ji]} = 0$, from (3) and (4), we have:

$$S_{jt} J_i^t - S_{it} J_j^t = \nabla_h \left(\nabla_j J_i^h - \nabla_i J_j^h \right) = 0.$$

Since $S_{ij} = S_{ji}$, from the last equation we have:

Theorem 3.1. *In an anti-Kähler–Codazzi manifold, the Ricci tensor is pure with respect to the complex structure J.*

We now put:

$$\overset{*}{S}_{ji} = -H_{jt} J_i^t = -R_{hjtl} G^{lh} J_i^t.$$

We call $\overset{*}{S}$ the Ricci* tensor of M. On the other hand, by virtue of $\overset{*}{S}_{jt} J_i^t = H_{ji}$ Eq. (4) can be written as:

$$\nabla_h \nabla_j J_i^h = S_{jt} J_i^t - \overset{*}{S}_{jt} J_i^t = \left(S_{jt} - \overset{*}{S}_{jt} \right) J_i^t.$$

Hence, we have

Theorem 3.2. *Let (M, g, J) be an anti-Kähler–Codazzi manifold. In order to have $S = \overset{*}{S}$, it is necessary and sufficient that:*

$$\nabla_h \nabla_j J_i^h = 0,$$

where S and $\overset{*}{S}$ are the Ricci and Ricci tensors, respectively.*

From this theorem, we have:

Corollary 3.3. *If an anti-Kähler–Codazzi manifold is anti-Kähler ($\nabla_j J_i^h = 0$), then $S = \overset{*}{S}$.*

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