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Partial Differential Equations

Uniqueness results for Stokes cascade systems and application to insensitizing controls

Resultats d'unicité pour des systèmes de Stokes en cascade et application au contrôle insensibilisant

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ABSTRACT

This Note is devoted to some insensitizing control problems for the Stokes system with a reduced number of controls. We give some ε-insensitization results with external unidirectional forces in different geometric configurations. We also provide a negative result of insensitization for the Stokes system in some 2-D manifold without boundary with one scalar control.

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RÉSUMÉ

Dans cette Note, on s'intéresse au problème d'insensibilisation pour le système de Stokes par un contrôle distribué unidirectionnel. On donne des résultats d'insensibilisation approchée avec un contrôle scalaire pour différentes configurations géométriques. D'autre part on donne un résultat négatif d'insensibilisation, par un contrôle scalaire, pour le système de Stokes posé sur une certaine variété bidimensionelle sans bord.

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Soit $\Omega \subset \mathbb{R}^3$ un ouvert connexe, ω et $\mathcal{O}$ deux ouverts non-vides inclus dans Ω, $\{e_i, i = 1, 2, 3\}$ la base canonique de $\mathbb{R}^3$ et $T > 0$. On considère le système de Stokes (2) avec données initiales incomplètes et un contrôle distribué $v\chi_\omega$ dans la direction $e \in \mathbb{R}^3$. On suppose que $y^0 \in \mathbb{H}$ est connu, $\widehat{y}^0 \in \mathbb{H}$ est *inconnu* avec $\|\widehat{y}^0\|_{L^2(\Omega)^3} = 1$ et τ est un réel petit ; où $\mathbb{H}$ est défini dans (1).

La notion de contrôle insensibilisant a été introduite dans [9] par J.-L. Lions, ainsi qu'une multitude de problèmes apparentés. On se donne une fonctionnelle J_τ d'observation de l'état du système définie la plupart du temps par (3) et il s'agit de trouver un contrôle v tel que les perturbations sur la donnée initiale n'affectent pas, du moins au premier ordre, cette observation. On parle alors de contrôle insensibilisant si on a (4) et de contrôle ε-insensibilisant si on a (5). En fait, dans le cas de la fonctionnelle (3) on prouve que l'existence d'un contrôle insensibilisant (resp. ε-insensibilisant) est équivalente à la contrôlabilité exacte (resp. approchée) à zéro du système en cascade (6), cf. [1] pour plus de détails.

L'existence de tels contrôles a été prouvée dans [7] pour la fonctionnelle (3). Plus récemment un résultat locale d'insensibilisation a été prouvé pour le système de Navier–Stokes dans [8]. Auparavant, l'existence de contrôles ε-insensibilisant pour (3) avait été obtenue dans [11] avec deux contrôles scalaire, enfin lorsque les zones de contrôle et d'observation ne se touchent pas, i.e. $\omega \cap \mathcal{O} = \emptyset$, ceci est fait dans [4].

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Le résultat principal de cette Note est donné dans le théorème suivant :

Théorème 0.1. *On suppose que* $\omega \cap \mathcal{O} \neq \emptyset$.

- *Si* $\Omega := G \times (0, L)$, *où* $G \subset \mathbb{R}^2$ *est régulier et* $L > 0$. *Alors, génériquement par rapport à G, la continuation unique* (10) *est vraie pour* $e \parallel e_3$.
- *Si* $\Omega := S \times \mathbb{R}$, *où S est un rectangle. Alors, la continuation unique* (10) *est encore vraie pour* $e \parallel e_1$ *ou* $e \parallel e_2$.

Autrement dit, sous les hypothèses ci-dessus, pour tout $\varepsilon > 0$ *et tout* $y^0 \in \mathbb{H}$, *il existe* $v \in L^2((0,T) \times \omega)^3$, *orienté dans la direction* e_i, *qui* ε*-insensibilise la fonctionnelle* (3).

La preuve est basée sur des résultats d'unicité désormais bien connus, cf. [6,10] et [12]. Soit $i \in \{1, 2, 3\}$. Dans un premier temps, on montre que $\varphi_i = 0$ dans $(0, T) \times \omega$ implique que $\psi_i = 0$ dans Q, en utilisant le fait que $\Delta\pi = 0$ dans $\mathcal{O} \times (0, T)$ et la propriété de prolongement unique parabolique (cf. [12]). Enfin, en utilisant [10] et [6], on en déduit que $\psi \equiv 0$ dans Q et, en particulier, $\psi(T, \cdot) = 0$ dans Ω.

Enfin, on s'intéresse à un exemple de donnée initiale qui ne peut être insensibilisée par un contrôle unidirectionnel pour le système de Stokes (14) posé sur le tore $\mathbb{T}_2$. On montre le résultat suivant :

Théorème 0.2. *Il existe* $y^0 \in L^2(\mathbb{T}_2)^2$, *tel que, pour tout* $v = (v_1, 0) \in L^2((0,T) \times \omega)^2$, *la fonctionnelle* J_τ, *avec* $\mathcal{O} = \mathbb{T}_2$, *ne peut être insensibilisée, même si* $\omega = \mathbb{T}_2$.

Comme précédemment l'existence d'un contrôle insensibilisant est équivalente à la contrôlabilité à zéro d'un système en cascade. La preuve repose sur la structure particulière du système en cascade (16) qui nous permet de faire des calculs explicites.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a connected open set. We denote by $\{e_i,\ i = 1, 2, 3\}$ the canonical basis of $\mathbb{R}^3$. For $T > 0$, we will use the notation $Q := (0, T) \times \Omega$ and $\Sigma := (0, T) \times \partial\Omega$. Let us recall the definition of some usual spaces in the context of incompressible fluids:

$$\mathbb{V} := \left\{ y \in H_0^1(\Omega)^3 \colon \nabla \cdot y = 0 \text{ in } \Omega \right\} \quad \text{and} \quad \mathbb{H} := \left\{ y \in L^2(\Omega)^3 \colon \nabla \cdot y = 0 \text{ in } \Omega,\ y \cdot n = 0 \text{ on } \partial\Omega \right\}. \tag{1}$$

• In the first part of this Note, we deal with the following Stokes control system with incomplete data:

$$\begin{cases} y_t - \Delta y + \nabla p = v \cdot e\chi_\omega,\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y^0 + \tau \widehat{y}^0 & \text{in } \Omega. \end{cases} \tag{2}$$

Here, $y : (0, T) \times \Omega \mapsto \mathbb{R}^3$ is the velocity of the particle of the fluid, v stands for the control function, e is a given vector of $\mathbb{R}^3$ which is the direction of the control and χ_ω is the characteristic function of an open set $\omega \subset \Omega$ where the control acts. The data is incomplete in the following sense: $y^0 \in \mathbb{H}$ is known, $\widehat{y}^0 \in \mathbb{H}$ is *unknown* with $\|\widehat{y}^0\|_{L^2(\Omega)^3} = 1$ and τ is a small *unknown* real number.

To be more precise about the problem under consideration, let $\mathcal{O}$ be a nonempty open set of Ω (called *observatory*). We introduce the following functional (called *sentinel*):

$$J_\tau(y) := \frac{1}{2} \iint_{\mathcal{O} \times (0,T)} |y|^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{3}$$

We say that the functional J_τ is *exactly insensitized* if there exists a control $v \in L^2((0, T) \times \omega)^3$ such that

$$\left. \frac{\mathrm{d}}{\mathrm{d}\tau} J_\tau(y) \right|_{\tau=0} = 0 \quad \forall \widehat{y}^0 \in L^2(\Omega)^3 \quad \text{such that } \left\| \widehat{y}^0 \right\|_{L^2(\Omega)^3} = 1. \tag{4}$$

The existence of such a control is proved in [7], for controls having all their components. In [2], this is achieved for the Navier–Stokes system with controls having one vanishing component. Here, we are interested in finding controls having two vanishing components. In this framework, in [10] and [6] the authors prove the existence of Lipschitz connected open subsets $\Omega \subset \mathbb{R}^3$ such that, even with $\omega = \Omega$, the null controllability of the Stokes system does not hold with two vanishing components for the control. Thus, as in [1] or [4] we are going to consider a relaxed version of the insensitivity property: given $\varepsilon > 0$, the control $v \in L^2((0, T) \times \omega)^3$ is said to ε*-insensitize* J_τ if

$$\left| \left. \frac{\mathrm{d}}{\mathrm{d}\tau} J_\tau(y) \right|_{\tau=0} \right| \leqslant \varepsilon \quad \forall \widehat{y}^0 \in L^2(\Omega)^3 \quad \text{such that } \left\| \widehat{y}^0 \right\|_{L^2(\Omega)^3} = 1. \tag{5}$$

In general, one can reformulate insensitivity conditions (4) and (5) in an equivalent way through a controllability problem for a cascade system (see [1]). More precisely, it suffices to introduce a formal adjoint of the equation satisfied by the derivative of y with respect to τ at $\tau = 0$ (which we call z):

$$\begin{cases} w_t - \Delta w + \nabla p^0 = v \cdot e \chi_\omega, \ \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + \nabla q = w \chi_{\mathcal{O}}, \ \nabla \cdot z = 0 & \text{in } Q, \\ w = z = 0 & \text{on } \Sigma, \\ w(0, \cdot) = y^0, \ z(T, \cdot) = 0 & \text{in } \Omega. \end{cases} \tag{6}$$

Here, (w, p^0) is the solution of system (2) for $\tau = 0$. In practice, we are led to prove partial-null (resp. partial-approximate) controllability for system (6), i.e. find v (resp. v_ε, $\forall \varepsilon > 0$) such that

$$z(0, \cdot) = 0 \quad \text{in } \Omega \ \big(\text{resp. } \big\|z(0, \cdot)\big\|_{L^2(\Omega)^3} \leqslant \varepsilon\big). \tag{7}$$

It is by now well known that the null (resp. approximate) controllability of a linear system is equivalent to an observability (resp. unique continuation) property for the adjoint system:

$$\begin{cases} -\varphi_t - \Delta \varphi + \nabla \pi = \psi \chi_{\mathcal{O}}, \ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \psi_t - \Delta \psi + \nabla \kappa = 0, \ \nabla \cdot \psi = 0 & \text{in } Q, \\ \varphi = \psi = 0 & \text{on } \Sigma, \\ \varphi(T, \cdot) = 0, \ \psi(0, \cdot) = \psi_0 & \text{in } \Omega. \end{cases} \tag{8}$$

More precisely, the partial-null controllability of system (6) is equivalent to the existence of $C > 0$ such that for all $\psi_0 \in \mathbb{H}$,

$$\big\|\psi(T, \cdot)\big\|_{L^2(\Omega)^3} \leqslant C \|\varphi \cdot e\|_{L^2((0,T)\times\omega)^3}, \tag{9}$$

and its partial-approximate controllability is equivalent to:

$$\varphi \cdot e \equiv 0 \quad \text{in } (0, T) \times \omega \quad \Longrightarrow \quad \psi(T, \cdot) = 0 \quad \text{in } \Omega. \tag{10}$$

• In the second part of this Note, we build an example where one cannot exactly insensitized the functional (3) for some initial data in $L^2(\Omega)$. For the heat equation in $1 - D$, the authors in [5] characterize the initial data for which J_τ can be exactly insensitized.

2. Insensitizing controls by unidirectional forces

In this section we assume that Ω is a three-dimensional cylinder, i.e., $\Omega = G \times (0, L)$ where $G \subset \mathbb{R}^2$ and $L > 0$. The first result we give here holds for a bounded smooth open subset G of $\mathbb{R}^2$ and a control direction e which is parallel to the cylinder generatrix ($e \parallel e_3$).

Theorem 2.1. *Assume that $\omega \cap \mathcal{O} \neq \emptyset$. Then, generically with respect to the cross section G, for all $T > 0$ and $L > 0$ the unique-continuation property* (10) *holds for $e \parallel e_3$, that is to say, J_τ can be approximately insensitized with controls of the form* $(0, 0, v_3)$.

Remark 1. More precisely, given any bounded domain G of $\mathbb{R}^2$ of class $\mathcal{C}^k$, with $k \geqslant 3$, we can find another domain $\widetilde{G}$ arbitrarily close to G in the $\mathcal{C}^k$ topology such that the unique continuation property (10) holds in $\widetilde{G}$ for $e \parallel e_3$.

Proof. Let $\mathcal{H} = \{y \in C^\infty(\overline{\Omega})^3;\ \nabla \cdot y = 0 \text{ in } \Omega,\ y \cdot n = 0 \text{ on } \partial\Omega\}$. Then, $\mathcal{H}$ is dense in $\mathbb{H}$. Thus, we can suppose without loss of generalities that ψ_0 lies in this space. Moreover, in virtue of the regularizing effect of Stokes like system (see, for instance, [13]), ψ and κ are smooth. We divide the proof in two steps:
Step 1. First we prove that, for the solutions of (8),

$$\varphi_3 \equiv 0 \quad \text{in } \omega \times (0, T) \quad \Longrightarrow \quad \psi_3 \equiv 0 \quad \text{in } Q. \tag{11}$$

Using the fact that $\Delta\pi = 0$ in $(0, T) \times \mathcal{O}$, $\Delta\varphi_3$ satisfies the following equation:

$$-(\Delta\varphi_3)_t - \Delta(\Delta\varphi_3) = \Delta\psi_3 \quad \text{in } (0, T) \times \mathcal{O}.$$

In particular, we have that $\Delta\psi_3 = 0$ in $(0, T) \times \mathcal{O} \cap \omega$.

We turn to the equation satisfied by $\Delta\psi_3$ in Q. Using now the fact that $\Delta\kappa = 0$ in Q, we obtain

$$(\Delta\psi_3)_t - \Delta(\Delta\psi_3) = 0 \quad \text{in } Q. \tag{12}$$

From the parabolic unique continuation property (see [12]) we deduce that $\Delta\psi_3 = 0$ in Q. This, together with the fact that $\psi_3 = 0$ on Σ gives us (11).
Step 2. We prove that

$$\psi_3 \equiv 0 \quad \text{in } \mathcal{O} \times (0, T) \quad \Longrightarrow \quad \psi \equiv 0 \quad \text{in } Q, \tag{13}$$

generically with respect to G. This result has been proved in [10, Theorem 1]. Indeed, they prove that (13) can be reduced to show that, generically with respect to the cross-section G, there is no eigenfunction of the Stokes system with third component identically zero. This is equivalent to prove that the eigenvalues of the Laplacian in $H_0^1(G)$ are simple. This property is well-known to be generically true with respect to G as long as G is regular, of class $\mathcal{C}^k$ ($k \geqslant 3$); this last property can be found in [14]. As a conclusion, $\psi(T,\cdot) = 0$ in Ω generically with respect to G. □

Remark 2. Observe that from the proof of Theorem 2.1 one can also establish that $\varphi \equiv 0$ in Q. Indeed, since $\varphi_3 = 0$ in $\omega \times (0,T)$ we have that $\partial_3 \pi = 0$ in $\omega \times (0,T)$ (recall that $\psi \equiv 0$ in Q). Therefore, by elliptic unique continuation $\partial_3 \pi \equiv 0$ in Q, so that

$$-\varphi_{3,t} - \Delta \varphi_3 = 0 \quad \text{in } Q.$$

Then, the parabolic unique continuation property and regularizing effect for Stokes like system yield that $\varphi_3 \equiv 0$ in Q. From step 2 above, we have again that $\varphi \equiv 0$ in Q generically with respect to G.

The second result we give holds in an unbounded open set:

$$\Omega := (S \times \mathbb{R}) \subset \mathbb{R}^3,$$

where S is a rectangle. This result is a direct consequence of [6].

Theorem 2.2. *Assume that $\omega \cap \mathcal{O} \neq \emptyset$. Then, for any rectangular cross-section S and for any $T > 0$ the unique continuation property (10) holds for $e \parallel e_i$, where $i \in \{1,2\}$. Consequently J_τ can be approximately insensitized with controls of the form $(v_1, 0, 0)$ or $(0, v_2, 0)$.*

Proof. The proof is very similar to the one of Theorem 2.1. Let $i \in \{1,2\}$. In a first step we prove that $\varphi_i = 0$ in $\omega \times (0,T)$ implies that $\psi_i = 0$ in Q, using again the fact that $\Delta \pi = 0$ in $\mathcal{O} \times (0,T)$ and the parabolic unique continuation property (see [12]). Then, it is proved in [6, Theorem 2] that $\psi \equiv 0$ in Q. □

3. Example of initial data which cannot be insensitized

Let $L_1 > 0$ and $L_2 > 0$ and let $\mathbb{T}_2$ be the flat torus $(\mathbb{R}/L_1\mathbb{Z}) \times (\mathbb{R}/L_2\mathbb{Z})$. Let ω be an open subset of $\mathbb{T}_2$. Set $Q := (0,T) \times \mathbb{T}_2$. Here, we consider the Stokes system with one distributed scalar control:

$$\begin{cases} y_t - \Delta y + \nabla p = \chi_\omega (v_1, 0) & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y(0,\cdot) = y^0 + \tau \widehat{y}^0 & \text{in } \mathbb{T}_2. \end{cases} \tag{14}$$

This system is far from being null controllable. Indeed, consider the closed linear subspace of $\mathbb{H}$:

$$\mathbb{H}_0 := \left\{ y \in \mathbb{H}\colon \int_{\mathbb{H}} y_2 = 0 \right\}. \tag{15}$$

Then, $\mathbb{H}_0$ is invariant i.e., for every $y(0,\cdot) \in \mathbb{H}_0$ and $v_1 \in L^2(Q)$, the solution of (14) is such that $y(t,\cdot) \in \mathbb{H}_0$ for all $t \in [0,T]$. Straightforward computations show that (14) is not null controllable in $\mathbb{H}_0$. Note that this invariance remains true for the Navier–Stokes system, see [3] for more details.

Our goal is to prove the following non-controllability result:

Theorem 3.1. *There exists $y^0 \in L^2(\mathbb{T}_2)^2$, such that for any $v_1 \in L^2((0,T) \times \omega)$ the functional J_τ, with $\mathcal{O} = \mathbb{T}_2$, cannot be insensitized even if $\omega = \mathbb{T}_2$.*

Proof. Without loss of generalities, y^0 can be taken in $\mathbb{H}_0$. Consider the functional given by (3) with $\mathcal{O} := \mathbb{T}_2$. Then, the insensitivity condition (4) is equivalent to $z(0,\cdot) = 0$ in $\mathbb{T}_2$, where z together with w solves the following cascade system:

$$\begin{cases} w_t - \Delta w + \nabla p^0 = \chi_\omega (v_1, 0),\ \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + \nabla q = w,\ \nabla \cdot z = 0 & \text{in } Q, \\ w(0,\cdot) = y^0,\ z(T,\cdot) = 0 & \text{in } \mathbb{T}_2. \end{cases} \tag{16}$$

Let $n \in \mathbb{Z}$, $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ and let $\zeta \in \mathcal{C}^\infty(\mathbb{T}_2)$ be defined by

$$\zeta(x_1, x_2) := \lambda_1 \sin\left(\frac{2n\pi x_1}{L_1}\right) + \lambda_2 \cos\left(\frac{2n\pi x_1}{L_1}\right), \quad \forall (x_1, x_2) \in \mathbb{T}_2. \tag{17}$$

Multiplying the second equations of (16) by ζ, integrating on $\mathbb{T}_2$ and integrating by parts leads to

$$\frac{d}{dt}\int_{\mathbb{T}_2} \zeta(x) w_2(x,t)\,dx = -\frac{4n^2\pi^2}{L_1^2}\int_{\mathbb{T}_2} \zeta(x) w_2(x,t)\,dx. \tag{18}$$

Multiplying the fourth equation of (16) by ζ, integrating on $\mathbb{T}_2$ and integrating by parts we obtain

$$-\frac{d}{dt}\int_{\mathbb{T}_2} \zeta(x) z_2(x,t)\,dx = -\frac{4n^2\pi^2}{L_1^2}\int_{\mathbb{T}_2} \zeta(x) z_2(x,t)\,dx + \int_{\mathbb{T}_2} \zeta(x) w_2(x,t)\,dx. \tag{19}$$

Taking into account that $z(\cdot,T)=0$ in $\mathbb{T}_2$ and $w(\cdot,0)=y^0$ in $\mathbb{T}_2$, we can explicitly integrate (18) and (19) to obtain

$$\int_{\mathbb{T}_2} \zeta(x) z_2(x,0)\,dx = \frac{L_1^2}{8n^2\pi^2}\big(1-e^{-\frac{8n^2\pi^2 T}{L_1^2}}\big)\int_{\mathbb{T}_2} \zeta(x) y_2^0(x)\,dx.$$

In particular, this proves that

$$\Big(\int_{\mathbb{T}_2} \zeta(x) y_2^0(x)\,dx \neq 0\Big) \quad\Longrightarrow\quad z_2(\cdot,0)\neq 0 \quad \text{in } \Omega. \qquad \square$$

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