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On the prescribed scalar curvature problem on S^n: The degree zero case

Sur le problème de courbure scalaire prescrite sur S^n : Le cas de degré zéro

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A R T I C L E   I N F O

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A B S T R A C T

In this Note, we consider the problem of the existence of conformal metrics with prescribed scalar curvature on the standard sphere S^n, $n \geqslant 3$. We give new existence and multiplicity results based on a new Euler–Hopf formula type. Our argument also has the advantage of extending the well known results due to Y. Li (1995) [10].

R É S U M É

Dans cette Note nous considérons le problème d'existence de métriques conformes avec courbure scalaire prescrite, sur la sphère standard S^n, $n \geqslant 3$. Nous donnons de nouveaux résultats d'existence et de multiplicité reposant sur un nouveau type de formule d'Euler–Hopf. Nos arguments ont également l'avantage d'étendre des résultats bien connus de Y. Li (1995) [10].

This Note is devoted to the problem of prescribing scalar curvature on the n-dimensional sphere $S^n, n \geqslant 3$. Let g_0 be the standard metric on S^n and $K : S^n \to \mathbb{R}$ be a smooth function. We are interested in the question whether there exists a metric g conformally equivalent to g_0 such that K is the scalar curvature of g. For $g = u^{\frac{4}{n-2}} g_0$, where u is a smooth positive function on S^n, the above question is equivalent to solving the following nonlinear PDE:

$$\begin{cases} -L_{g_0} u = K u^{\frac{n+2}{n-2}}, \\ u > 0, \quad \text{on } S^n, \end{cases} \tag{1}$$

where $L_{g_0} = -\frac{4(n-1)}{n-2} \triangle_{g_0} + n(n-1)$ is the conformal Laplacian operator of (S^n, g_0).

Problem (1) has a variational structure. A natural space to look at for solutions is $H^1(S^n)$. In virtue of the noncompactness of the injection $H^1(S^n) \hookrightarrow L^{\frac{2n}{n-2}}(S^n)$, the Euler functional associated to (1) does not satisfy the Palais–Smale condition, and that leads to the failure of the standard critical point theory. Moreover, besides the obvious necessary condition that K must be positive somewhere, there is a topological obstruction to solve (1), called the Kazdan–Warner obstruction, see [9].

The problem of prescribing scalar curvature has been widely studied, see for example the monographs [3,4] and the references therein.

One group of existence results to problem (1) has been obtained under the following β-flatness condition:

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$(f)_\beta$ Assume that $K : S^n \to \mathbb{R}$, $n \geqslant 3$ be a C^1 function such that for any critical point y of K, there exists some real number $\beta = \beta(y)$, such that in some geodesic normal coordinate system centered at y, we have

$$K(x) = K(y) + \sum_{k=1}^{n} b_k \left|(x-y)_k\right|^\beta + R(x-y),$$

where $b_k = b_k(y) \neq 0$, $\forall k = 1, \dots, n$, $\sum_{k=1}^n b_k \neq 0$ and $\lim_{|z|\to 0} \sum_{s=0}^{[\beta]} |\nabla^s R(z)||z|^{-\beta+s} = 0$.

Here ∇^s denotes all possible derivatives of order s and $[\beta]$ is the integer part of β.

A typical existence result says that a solution of (1) exists provided K is a positive function satisfying $(f)_\beta$ and

$$\sum_{y\in\mathcal{K}^+} (-1)^{n-\widetilde{i}(y)} \neq 1, \tag{R_1}$$

where $\mathcal{K}^+ = \{y \in S^n,\ \nabla K(y) = 0 \text{ and } -\sum_{k=1}^n b_k > 0\}$ and $\widetilde{i}(y) = \sharp\{b_k = b_k(y), 1 \leqslant k \leqslant n \text{ s.t. } b_k < 0\}$. For $n-2 < \beta < n$ this result has been given by Y. Li [10] (see also [11], where the case of $\beta = n-2$ is handled under some further condition on the curvature K). For $1 < \beta < n$, the result has been given by Y. Li [10] and Ambrosetti, Garcia and Peral [2] under a perturbation condition, that is for K close to a constant function. The method used in [10] and [11] is based on a fine blow-up analysis of some subcritical approximations and the use of the topological degree tools, while the approach of [2] relies on an abstract perturbation method due to [1].

A natural question that arises when looking at the above results is what happens when the total sum in (R_1) is equal to 1, but a partial sum is not. Under which condition can one use this partial sum to derive existence results? Our aim in this work is to give a partial answer to this question and to give new existence results which generalize the previous existence results obtained in [10] and [2]. Moreover, in generic cases, we give a lower bound of the number of conformal metrics with prescribed scalar curvature K. To state our results, we consider the following assumption: we say that an integer $k \in (\mathrm{A}_1)$ if it satisfies

(A_1) For each $y \in \mathcal{K}^+$, we have $n - \widetilde{i}(y) \neq k+1$.

Our first main result can be stated as follows:

Theorem 1. *Let K be a positive function satisfying $(f)_\beta$ with $n-2 < \beta < n$. If*

$$\max_{k\in(\mathrm{A}_1)} \left| 1 - \sum_{\substack{y\in\mathcal{K}^+,\\ n-\widetilde{i}(y)\leqslant k}} (-1)^{n-\widetilde{i}(y)} \right| \neq 0,$$

then there exists a solution to problem (1). Moreover, for generic K, one has

$$\sharp S \geqslant \max_{k\in(\mathrm{A}_1)} \left| 1 - \sum_{\substack{y\in\mathcal{K}^+,\\ n-\widetilde{i}(y)\leqslant k}} (-1)^{n-\widetilde{i}(y)} \right|,$$

where S denotes the set of solutions of (1).

The reader should observe that any integer $k \geqslant n$ satisfies condition (A_1). Thus, as a consequence of the above theorem, we have the following corollary, which recovers a previous existence result of [10]:

Corollary 1. *Assume that K is a positive function satisfying $(f)_\beta$ for $n-2 < \beta < n$. If*

$$\sum_{y\in\mathcal{K}^+} (-1)^{n-\widetilde{i}(y)} \neq 1,$$

then problem (1) has at least one solution. Moreover, for generic K, one has

$$\sharp S \geqslant \left| 1 - \sum_{y\in\mathcal{K}^+} (-1)^{n-\widetilde{i}(y)} \right|.$$

In the second part of this work, we consider the perturbative case, that is, the case where the prescribed scalar curvature is $K = 1 + \varepsilon K_0$, for $K_0 \in C^1(S^n)$ and $|\varepsilon|$ small. Then we are reduced to study the problem

$$-L_{g_0} u = (1+\varepsilon K_0) u^{\frac{n+2}{n-2}}, \qquad u > 0 \quad \text{on } S^n. \tag{P_ε}$$

Problem (P_ε) was studied first by Chang and Yang [8], under a suitable nondegeneracy condition and later by Li [10] and Ambrosetti, Garcia and Peral [2] under a β flatness condition.

Our aim is to tackle problem (P_ε), using another approach different from the ones used in the above mentioned papers, and to extend the existence result considered in Theorem 1 by allowing any $\beta \in]1, n[$. More precisely, we will prove the following:

Theorem 2. *Let K be a positive function satisfying $(f)_\beta$ with $1 < \beta < n$. If*

$$\max_{k\in(\mathrm{A}_1)}\left|1 - \sum_{\substack{y\in\mathcal{K}^+,\\ n-\widetilde{i}(y)\leqslant k}} (-1)^{n-\widetilde{i}(y)}\right| \neq 0,$$

then for $|\varepsilon|$ sufficiently small, there exists a solution to problem (P_ε).

Moreover, for generic K, if $\frac{n-2}{2} < \beta < n$, one has

$$\sharp S \geqslant \max_{k\in(\mathrm{A}_1)}\left|1 - \sum_{\substack{y\in\mathcal{K}^+,\\ n-\widetilde{i}(y)\leqslant k}} (-1)^{n-\widetilde{i}(y)}\right|,$$

where S denotes the set of solutions of (P_ε).

The following result, proved by Y. Li [10], is an immediate corollary of Theorem 2:

Corollary 2. *(See [10].) Assume that K is a positive function satisfying $(f)_\beta$, $1 < \beta < n$. If*

$$\sum_{y\in\mathcal{K}^+} (-1)^{n-\widetilde{i}(y)} \neq 1,$$

then for $|\varepsilon|$ sufficiently small, (P_ε) has at least one solution.

A generalization of Corollary 2 using the degree of the related function has been proved by Ambrosetti, Garcia and Peral [2]. Such a result extends the result of Corollary 2 to positives functions satisfying $(f)_\beta$ with $b_k = b_k(y) \in \mathbb{R}$, $k = 1, \ldots, n$. This degree actually computes the Leray–Schauder degree of (P_ε). In the special case where $b_k = b_k(y) \in \mathbb{R}^\star$ for any $k = 1, \ldots, n$, this degree can be expressed as

$$d = 1 - \sum_{y\in\mathcal{K}^+} (-1)^{n-\widetilde{i}(y)}.$$

Therefore, for positive functions K satisfying $(f)_\beta$, Theorem 2 recovers also the existence result of [2].

In the sequel, we give a brief description of the main ingredients behind the proof of our results. Our approach is completely different from the one used in [10] and [2]. Our argument uses a careful analysis of the loss of compactness of the associated Euler–Lagrange functional J. Namely, we characterize the noncompact orbits of the gradient flow of J, the so-called critical points at infinity, following the terminology of A. Bahri [3]. This characterization is obtained through the construction of a suitable pseudogradient in the set of potential critical points at infinity $V(p, \varepsilon, \omega)$, $p \in \mathbb{N}^\star$, $\varepsilon > 0$ small enough and where ω is either a solution of (1) or zero (see [4]).

In the case where $\omega \neq 0$, along the flow lines of such a pseudogradient, the interaction between the solution ω and the bubbles $\delta_{(a_i,\lambda_i)}$, $i = 1, \ldots, p$ given by the term $\frac{w(a_i)}{\lambda_i^{\frac{n-2}{2}}}$ is large and dominates the self-interaction of $\delta_{(a_i,\lambda_i)}$ given by the term $\frac{1}{\lambda_i^\beta}$. This phenomenon appears only for $\frac{n-2}{2} < \beta$. Such an effect implies that the Palais–Smale condition is satisfied along the decreasing flow lines, so these flow lines will escape the set $V(p, \varepsilon, \omega)$ and hence no critical point at infinity occurs in this case.

In the case where $\omega = 0$, we observe that the interaction between two different bubbles $\delta_{(a_i,\lambda_i)}$ and $\delta_{(a_j,\lambda_j)}$, with $\lambda_i \sim \lambda_j$ given by the term $\frac{G(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}$ is large and dominates the self-interaction given by the term $\frac{1}{\lambda_i^\beta}$; this phenomenon appears only for $\beta > n-2$. Using the same argument as in the above case, we rule out the existence of critical points at infinity in $V(p, \varepsilon) := V(p, \varepsilon, 0)$ for $p \geqslant 2$. Concerning the left case, when $p = 1$, we claim that the Palais–Smale condition is satisfied along the decreasing flow lines in $V(1, \varepsilon)$ as long as these flow lines do not enter the neighborhood of a critical point y of K such that $y \in \mathcal{K}^+$, and thus the critical points at infinity are in one to one correspondence with the critical points y of K such that $y \in \mathcal{K}^+$. These critical points at infinity can be treated as usual critical points once the Morse lemma at infinity is invoked, from which we can derive just as in the classical Morse theory the difference of topology induced by these critical points at infinity and compute their Morse indices. A topological argument similar to the one in [5] and [6] allows us to derive our results. The detail of the proofs will be given in [7].

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References

[1] A. Ambrosetti, M. Badiale, Homoclinics: Poincaré–Melnikov type results via variational approach, Ann. Inst. H. Poincaré Anal. Non Linéaire 15 (1998) 233–252.
[2] A. Ambrosetti, J. Garcia Azorero, A. Peral, Perturbation of $-\Delta u + u^{\frac{(N+2)}{(N-2)}} = 0$, the scalar curvature problem in $\mathbb{R}^N$ and related topics, J. Funct. Anal. 165 (1999) 117–149.
[3] A. Bahri, Critical Point at Infinity in Some Variational Problems, Pitman Res. Notes Math., vol. 182, Longman Sci. Tech., Harlow, 1989.
[4] A. Bahri, An invariant for Yamabe-type flows with applications to scalar curvature problems in high dimensions, in: A Celebration of J.F. Nash Jr., Duke Math. J. 81 (1996) 323–466.
[5] R. Ben Mahmoud, H. Chtioui, Existence results for the prescribed scalar curvature on $\mathbb{S}^3$, Ann. Inst. Fourier 61 (2011) 971–986.
[6] R. Ben Mahmoud, H. Chtioui, Prescribing the scalar curvature problem on higher-dimensional manifolds, Discrete Contin. Dyn. Syst. A 32 (5) (2012) 1857–1879.
[7] R. Ben Mahmoud, H. Chtioui, A. Rigane, On the prescribed scalar curvature on S^n: The degree zero case, in preparation.
[8] A. Chang, P. Yang, A perturbation result in prescribing scalar curvature on S^n, Duke Math. J. 64 (1991) 27–69.
[9] J. Kazdan, J. Warner, Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature, Ann. Math. 101 (1975) 317–331.
[10] Y.Y. Li, Prescribing scalar curvature on S^n and related topics, Part I, J. Diff. Eq. 120 (1995) 319–410.
[11] Y.Y. Li, Prescribing scalar curvature on S^n and related topics, Part II: Existence and compactness, Comm. Pure Appl. Math. 49 (1996) 541–579.