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Null controllability of a degenerated reaction–diffusion system in cardiac electro-physiology

Nulle contrôlabilité d'un système dégénéré de réaction–diffusion dans l'électique-physiologie du cœur

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ABSTRACT

This Note is devoted to the analysis of the null controllability of a nonlinear reaction–diffusion system, approximating a parabolic–elliptic system, modeling electrical activity in the heart. The uniform, with respect to the degenerating parameter, null controllability of the approximating system by a single control force acting on a subdomain is shown. The proof needs a precise estimate with respect to the degenerating parameter and it is done combining Carleman estimates and energy inequalities.

RÉSUMÉ

Ce travail est consacré à l'analyse de la nulle contrôlabilité d'un système dégénéré de réaction–diffusion non-linéaire modélisant l'activité électrique du cœur. Notre contrôle agit dans un sous-domaine fixe du domaine du coeur. Nous prouvons la nulle contrôlabilité de notre modèle en établissant en particulier une estimation de Carleman pour l'équation dégénéré. Des estimations globales de type Carleman et la régularité parabolique sont employées.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough and ω a nonempty subset of Ω. Given $T > 0$, in Ω we consider the controllable *bidomain model* (in $\mathbb{R}^3$ it describes electrical activities):

$$\begin{cases} c_m \partial_t v - \operatorname{div}\big(\mathbf{M}_i(x)\nabla u_i\big) + h(v) = f\chi_\omega & \text{in } Q := \Omega \times (0,T), \\ c_m \partial_t v + \operatorname{div}\big(\mathbf{M}_e(x)\nabla u_e\big) + h(v) = 0 & \text{in } Q := \Omega \times (0,T), \\ u_i = 0,\ u_e = 0 & \text{on } \Sigma := \partial\Omega \times (0,T), \\ v(0) = v_0 & \text{in } \Omega. \end{cases} \tag{1}$$

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In (1), $v = u_i - u_e$, h is a nonlinear function, χ_ω is the characteristic function of ω and f is a control. We assume that the diffusion matrices M_i and M_e are of class C^∞ and uniformly elliptic in x. If $\mathbf{M}_i \equiv \mu \mathbf{M}_e$ for some constant $\mu \in \mathbb{R}$, then (1) is simplified to the *monodomain model*

$$\begin{cases} c_m \partial_t v - \frac{\mu}{\mu+1} \operatorname{div}\big(\mathbf{M}_e(x)\nabla v\big) + h(v) = f\chi_\omega & \text{in } Q, \\ -\operatorname{div}\big(\mathbf{M}(x)\nabla u_e\big) = \operatorname{div}\big(\mathbf{M}_i(x)\nabla v\big) & \text{in } Q, \\ v = 0,\ u_e = 0 & \text{on } \Sigma, \\ v(0) = v_0 & \text{in } \Omega, \end{cases} \tag{2}$$

where $\mathbf{M} = \mathbf{M}_e + \mathbf{M}_i$.

We approximate the monodomain model by the family of parabolic systems

$$\begin{cases} c_m \partial_t v^\epsilon - \frac{\mu}{\mu+1} \operatorname{div}\big(\mathbf{M}_e(x)\nabla v^\epsilon\big) + h\big(v^\epsilon\big) = f^\epsilon\chi_\omega & \text{in } Q, \\ \varepsilon\partial_t u_e^\epsilon - \operatorname{div}\big(\mathbf{M}(x)\nabla u_e^\epsilon\big) = \operatorname{div}\big(\mathbf{M}_i(x)\nabla v^\epsilon\big) & \text{in } Q, \\ v^\epsilon = 0,\ u_e^\epsilon = 0 & \text{on } \Sigma, \\ v^\epsilon(0) = v_0,\ u_e^\epsilon(0) = u_{e,0} & \text{in } \Omega, \end{cases} \tag{3}$$

where ϵ is a positive constant and $u_{e,0}$ is any function in the space of initial data.

The purpose of this Note is to give an answer to the following question:

If there exists a control f^ϵ, for each $\epsilon > 0$, that drives the solution $(v^\epsilon, u_e^\epsilon)$ of (3) to zero at time $t = T$, i.e.

$$v^\epsilon(T) = u_e^\epsilon(T) = 0,$$

is it true that when $\epsilon \to 0$ the sequence of controls f^ϵ converges to a function f that drives the solution (v, u_e) of (2) to zero at time $t = T$?

Since the bidomain model is a system of two coupled parabolic equations and the monodomain model is a system of parabolic–elliptic type, these two systems have, at least a priori, different control properties. Therefore, it is natural to ask if the controllability of the monodomain model can be seen as a limit process of the controllability of a family of parabolic systems.

It is not difficult to see that a positive answer to this question is equivalent with proving that the control sequence f^ϵ is bounded with respect to ϵ.

In order to answer the question made before, we consider the following linearized version of (3):

$$\begin{cases} c_m \partial_t v^\epsilon - \frac{\mu}{\mu+1} \operatorname{div}\big(\mathbf{M}_e(x)\nabla v^\epsilon\big) + a(x,t)v^\epsilon = f^\epsilon\chi_\omega & \text{in } Q, \\ \varepsilon\partial_t u_e^\epsilon - \operatorname{div}\big(\mathbf{M}(x)\nabla u_e^\epsilon\big) = \operatorname{div}\big(\mathbf{M}_i(x)\nabla v^\epsilon\big) & \text{in } Q, \\ v^\epsilon = 0,\ u_e^\epsilon = 0 & \text{on } \Sigma, \\ v^\epsilon(0) = v_0,\ u_e^\epsilon(0) = u_{e,0} & \text{in } \Omega, \end{cases} \tag{4}$$

for a given function $a = a(x,t)$ in $L^\infty(Q)$.

Our first main result in this work gives the uniform null controllability of (4).

Theorem 1. *Given v_0 and $u_{e,0}$ in $L^2(\Omega)$, then, for each $\varepsilon > 0$, there exists a control $f^\epsilon \in L^2(\omega \times (0,T))$ so that the solution $(v^\epsilon, u_e^\epsilon)$ of (4) is driven to zero at time T, i.e.*

$$v^\epsilon(T) = 0,\ u_e^\epsilon(T) = 0.$$

Moreover, the control f^ϵ satisfies

$$\big\|f^\epsilon\chi_\omega\big\|^2_{L^2(Q)} \leqslant C\big(\|v_0\|^2_{L^2(\Omega)} + \varepsilon\|u_{e,0}\|^2_{L^2(\Omega)}\big). \tag{5}$$

The next second main result gives a positive answer to the question made above.

Theorem 2. *Given v_0 and $u_{e,0}$ in $L^2(\Omega)$ and let q_N satisfying*

$$q_N \in (2,\infty) \quad \text{if } N = 1,2,\ \frac{N+2}{2} < q_N < 2\frac{N+2}{N-2} \text{ if } N \geqslant 3. \tag{6}$$

We have:

• *If h is $C^1(\mathbb{R})$, global Lipschitz and satisfies $h(0) = 0$. There exists a control $f^\epsilon \in L^2(\omega \times (0,T))$ such that the solution $(v^\varepsilon, u_e^\varepsilon)$ of (3) satisfies*

$$v^\varepsilon(T) = u_e^\varepsilon(T) = 0.$$

Besides, the control f^ε has the estimate

$$\big\|f^\varepsilon\chi_\omega\big\|_{L^2(Q)} \leqslant C\big(\|v_0\|_{L^2(\Omega)} + \varepsilon\|u_{e,0}\|_{L^2(\Omega)}\big). \tag{7}$$

• *If h is a C^1 function satisfying*

$$h(0)=0, \qquad \frac{h(v_1)-h(v_2)}{v_1-v_2} \geqslant -C, \ \forall v_1 \neq v_2, \tag{8}$$

$$0 < \liminf_{|v|\to\infty} \frac{h(v)}{v^3} \leqslant \limsup_{|v|\to\infty} \frac{h(v)}{v^3} < \infty \tag{9}$$

and $(v_0, u_{e,0}) \in (H_0^1(\Omega) \cap W^{2(1-\frac{1}{q_N}),q_N}(\Omega))^2$, *with* $\|(v_0, u_{e,0})\|_{L^\infty(\Omega)} \leqslant \gamma$, *for a sufficient small* $\gamma > 0$ *does not depending on* ε. *There exists a control* $f^\varepsilon \in L^{q_N}(\omega \times (0,T))$ *such that the solution* $(v^\varepsilon, u_e^\varepsilon)$ *of* (3), *with* $(v^\varepsilon, u_e^\varepsilon) \in (W^{2,1}_{q_N}(Q))^2$, *satisfies*

$$v^\varepsilon(T) = u_e^\varepsilon(T) = 0.$$

Moreover, the control f^ϵ *has the estimate*

$$\|f^\epsilon \chi_\omega\|^2_{L^{q_N}(Q)} \leqslant C\big(\|v_0\|^2_{L^2(\Omega)} + \epsilon\|u_{e,0}\|^2_{L^2(\Omega)}\big). \tag{10}$$

Proof of Theorem 2 is done combining Theorem 1 and an appropriate version of Kakutani's fixed point theorem (see, for example, [4]) and will not be reproduced here. In the next section we will focus in the proof of Theorem 1. The detailed proofs will be given in a forthcoming paper.

2. Uniform null controllability for the linearized system

It is well known that the proof of Theorem 1 is equivalent with proving an observability inequality for the adjoint system:

$$\begin{cases} -c_m\partial_t\varphi^\epsilon - \frac{\mu}{\mu+1}\operatorname{div}\big(M_e(x)\nabla\varphi^\epsilon\big) + a(t,x)\varphi^\epsilon = \operatorname{div}\big(M_i(x)\nabla\varphi_e^\epsilon\big) & \text{in } Q, \\ -\varepsilon\partial_t\varphi_e^\epsilon - \operatorname{div}\big(M(x)\nabla\varphi_e^\epsilon\big) = 0 & \text{in } Q, \\ \varphi^\epsilon = 0, \ \varphi_e^\epsilon = 0 & \text{on } \Sigma, \\ \varphi^\epsilon(T) = \varphi_T, \ \varphi_e^\epsilon(T) = \varphi_{e,T} & \text{in } \Omega. \end{cases} \tag{11}$$

More precisely, in order to prove Theorem 1, it is sufficient to show the existence of an universal constant C, which is bounded with respect to ϵ, so that the observability inequality

$$\varepsilon\big\|\varphi_e^\epsilon(0)\big\|^2_{L^2(\Omega)} + \big\|\varphi^\epsilon(0)\big\|^2_{L^2(\Omega)} \leqslant C \iint_{\omega\times(0,T)} \big|\varphi^\epsilon\big|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{12}$$

holds for every solution of the adjoint system (11) with initial data $(\varphi_T, \varphi_{e,T}) \in L^2(\Omega)^2$.

To prove (12), we consider φ_T and $\varphi_{e,T}$ smooth enough and define $\rho^\epsilon(x,t) = \operatorname{div}(M(x)\nabla\varphi_e^\epsilon(x,t))$. The pair $(\varphi^\epsilon, \rho^\epsilon)$ satisfies:

$$\begin{cases} -c_m\partial_t\varphi^\epsilon - \frac{\mu}{\mu+1}\operatorname{div}\big(M_e(x)\nabla\varphi^\epsilon\big) + a(x,t)\varphi^\epsilon = \frac{\mu}{\mu+1}\rho^\epsilon & \text{in } Q, \\ -\varepsilon\partial_t\rho^\epsilon - \operatorname{div}\big(M(x)\nabla\rho^\epsilon\big) = 0 & \text{in } Q, \\ \varphi^\epsilon = 0, \ \rho^\epsilon = 0 & \text{on } \Sigma, \\ \varphi^\epsilon(T) = \varphi_T, \ \rho^\epsilon(T) = \rho_T & \text{in } \Omega. \end{cases} \tag{13}$$

We apply a Carleman estimate for non-degenerate heat equations to Eq. $(13)_1$ (see, for example, [2,3,5–7]) and apply a sharp Carleman inequality, with respect to ϵ (the proof can be found in [1]), to Eq. $(13)_2$. Combining this two inequalities we are able to obtain a Carleman type estimate in the form

$$\iint_Q \beta_1^2\big|\varphi^\epsilon\big|^2 \,\mathrm{d}x\,\mathrm{d}t + \iint_Q \beta_2^2\big|\rho^\epsilon\big|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C\varepsilon^{-2} \iint_{\omega\times(0,T)} \beta_3^2|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{14}$$

for some appropriate weight functions $\beta_i := \beta_i(x,t)$ (for $i = 1,2,3$) and some constant $C = C(\Omega, \omega, \|a\|_\infty, T) > 0$.

Next, we get rid of the term ϵ^{-2} appearing in the right-hand side of (14). For that, we take a weight function $\beta_4 = \beta_4(t)$ satisfying $|(\beta_4)_t(t)| \leqslant C\beta_2(x,t)$ for all $(x,t) \in Q$ and we show that

$$\iint_Q \beta_4^2\big|\rho^\epsilon\big|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C\varepsilon^2 \iint_Q \beta_2^2\big|\rho^\epsilon\big|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{15}$$

Inequality (15) is proved applying an energy inequality for the heat like equation satisfied by $\beta_4\rho^\epsilon$.
We combine (14) and (15) in order to get

$$\iint_Q \beta_4^2 |\rho^\epsilon|^2 \,dx\,dt \leqslant C \iint_{\omega\times(0,T)} \beta_3^2 |\varphi^\epsilon|^2 \,dx\,dt. \tag{16}$$

Using (16) and a Carleman estimate to $(13)_1$ we show that

$$\iint_Q \beta_5^2 |\varphi^\epsilon|^2 \,dx\,dt \leqslant C \iint_{\omega\times(0,T)} \beta_3^2 |\varphi^\epsilon|^2 \,dx\,dt, \tag{17}$$

for some appropriate weight function $\beta_5 := \beta_5(x,t)$.

Putting (16) and (17) together, we obtain

$$\iint_Q \beta_5^2 |\varphi^\epsilon|^2 \,dx\,dt + \iint_Q \beta_4^2 |\rho^\epsilon|^2 \,dx\,dt \leqslant C \iint_{\omega\times(0,T)} \beta_3^2 |\varphi|^2 \,dx\,dt. \tag{18}$$

Using (18) and energy estimates, it is not difficult to show that

$$\|\varphi^\epsilon(0)\|^2_{L^2(\Omega)} + \epsilon \|\rho^\epsilon(0)\|^2_{L^2(\Omega)} \leqslant C \iint_{\omega\times(0,T)} \beta_3 |\varphi^\epsilon|^2 \,dx\,dt, \tag{19}$$

for some constant $C = C(\Omega, \omega, \|a\|_\infty, T) > 0$ and it follows from the definition of ρ^ϵ that

$$\|\varphi^\epsilon(0)\|^2_{L^2(\Omega)} + \epsilon \|\varphi_e^\epsilon(0)\|^2_{L^2(\Omega)} \leqslant C \iint_{\omega\times(0,T)} |\varphi^\epsilon|^2 \,dx\,dt, \tag{20}$$

which is the observability inequality (12) in the case where we have smooth solutions.

From the density of smooth solutions in the space where the solutions of (11) live, we conclude that the observability inequality is satisfied by all solutions of (11). This concludes the proof of Theorem 1.

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