C. R. Acad. Sci. Paris, Ser. I 350 (2012) 553–556

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Algebra/Lie Algebras

Every monomorphism of the Lie algebra of triangular polynomial derivations is an automorphism

Tout homomorphisme injectif de l'algèbre de Lie des dérivations triangulaires polynomiales est un automorphisme

Vladimir V. Bavula

Department of Pure Mathematics, University of Sheffield, Hicks Building, Sheffield S3 7RH, UK

A R T I C L E   I N F O

Article history:
Received 23 May 2012
Accepted 8 June 2012
Available online 28 June 2012

Presented by the Editorial Board

A B S T R A C T

We prove that every monomorphism of the Lie algebra $\mathfrak{u}_n$ of triangular derivations of the polynomial algebra $P_n = K[x_1, \dots, x_n]$ is an automorphism.

R É S U M É

Nous montrons que tout homomorphisme injectif de l'algèbre de Lie $\mathfrak{u}_n$ des dérivations triangulaires de l'algèbre de polynômes $P_n = K[x_1, \dots, x_n]$ est un automorphisme.

1. Introduction

Throughout, K is a field of characteristic zero and K^* is its group of units; $P_n := K[x_1, \dots, x_n] = \bigoplus_{\alpha \in \mathbb{N}^n} Kx^\alpha$ is a polynomial algebra over K where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$; $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives (K-linear derivations) of P_n; $\mathrm{Aut}_K(P_n)$ is the group of automorphisms of the polynomial algebra P_n; $\mathrm{Der}_K(P_n) = \bigoplus_{i=1}^n P_n \partial_i$ is the Lie algebra of K-derivations of P_n; $A_n := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} Kx^\alpha \partial^\beta$ is the nth *Weyl algebra*; for each natural number $n \geqslant 2$,

$$\mathfrak{u}_n := K\partial_1 + P_1\partial_2 + \cdots + P_{n-1}\partial_n$$

is the *Lie algebra of unitriangular polynomial derivations* (it is a Lie subalgebra of the Lie algebra $\mathrm{Der}_K(P_n)$) and $G_n := \mathrm{Aut}_K(\mathfrak{u}_n)$ is its group of automorphisms; $\delta_1 := \mathrm{ad}(\partial_1), \dots, \delta_n := \mathrm{ad}(\partial_n)$ are the inner derivations of the Lie algebra $\mathfrak{u}_n$ determined by the elements $\partial_1, \dots, \partial_n$ (where $\mathrm{ad}(a)(b) := [a, b]$).

The aim of the Note is to prove the following theorem:

Theorem 1.1. *Every monomorphism of the Lie algebra $\mathfrak{u}_n$ is an automorphism.*

Remark. Not every epimorphism of the Lie algebra $\mathfrak{u}_n$ is an automorphism. Moreover, there are countably many distinct ideals $\{I_{i\omega^{n-1}} \mid i \geqslant 0\}$ such that

E-mail address: v.bavula@sheffield.ac.uk.

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.06.001

$$I_0 = \{0\} \subset I_{\omega^{n-1}} \subset I_{2\omega^{n-1}} \subset \cdots \subset I_{i\omega^{n-1}} \subset \cdots$$

and the Lie algebras $\mathfrak{u}_n / I_{i\omega^{n-1}}$ and $\mathfrak{u}_n$ are isomorphic [3, Theorem 5.1(1)].

Theorem 1.1 has bearing of the Conjecture of Dixmier [6] for the Weyl algebra A_n over a field of characteristic zero that claims: *every homomorphism of the Weyl algebra is an automorphism*. The Weyl algebra A_n is a simple algebra, so every algebra endomorphism of A_n is a monomorphism. This conjecture is open since 1968 for all $n \geqslant 1$. It is stably equivalent to the Jacobian Conjecture for the polynomial algebras as was shown by Tsuchimoto [7], Belov-Kanel and Kontsevich [5] (see also [1] for a short proof). The Jacobian Conjecture claims that *certain* monomorphisms of the polynomial algebra P_n are isomorphisms: *Every algebra endomorphism σ of the polynomial algebra P_n such that $\mathcal{J}(\sigma) := \det(\frac{\partial \sigma(x_i)}{\partial x_j}) \in K^*$ is an automorphism.* The condition that $\mathcal{J}(\sigma) \in K^*$ implies that the endomorphism σ is a monomorphism.

An analogue of the Conjecture of Dixmier is true for the algebra $\mathbb{I}_1 := K\langle x, \frac{\mathrm{d}}{\mathrm{d}x}, \int \rangle$ of polynomial integro-differential operators.

Theorem 1.2. *(See [2, Theorem 1.1].) Each algebra endomorphism of $\mathbb{I}_1$ is an automorphism.*

In contrast to the Weyl algebra $A_1 = K\langle x, \frac{\mathrm{d}}{\mathrm{d}x} \rangle$, the algebra of polynomial differential operators, the algebra $\mathbb{I}_1$ is neither a left/right Noetherian algebra nor a simple algebra. The left localizations, $A_{1,\partial}$ and $\mathbb{I}_{1,\partial}$, of the algebras A_1 and $\mathbb{I}_1$ at the powers of the element $\partial = \frac{\mathrm{d}}{\mathrm{d}x}$ are isomorphic. For the simple algebra $A_{1,\partial} \simeq \mathbb{I}_{1,\partial}$, there are algebra endomorphisms that are not automorphisms [2].

Before giving the proof of Theorem 1.1, let us recall several results that are used in the proof.

1.1. *The derived series for the Lie algebra $\mathfrak{u}_n$*

Let $\mathcal{G}$ be a Lie algebra over the field K and $\mathfrak{a}$, $\mathfrak{b}$ be its ideals. The *commutant* $[\mathfrak{a}, \mathfrak{b}]$ of the ideals $\mathfrak{a}$ and $\mathfrak{b}$ is the linear span in $\mathcal{G}$ of all the elements $[a, b]$ where $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Let $\mathcal{G}_{(0)} := \mathcal{G}$, $\mathcal{G}_{(1)} := [\mathcal{G}, \mathcal{G}]$ and $\mathcal{G}_{(i)} := [\mathcal{G}_{(i-1)}, \mathcal{G}_{(i-1)}]$ for $i \geqslant 2$. The descending series of ideals of the Lie algebra $\mathcal{G}$,

$$\mathcal{G}_{(0)} = \mathcal{G} \supseteq \mathcal{G}_{(1)} \supseteq \cdots \supseteq \mathcal{G}_{(i)} \supseteq \mathcal{G}_{(i+1)} \supseteq \cdots$$

is called the *derived series* for the Lie algebra $\mathcal{G}$. The Lie algebra $\mathfrak{u}_n$ admits the finite strictly descending chain of ideals

$$\mathfrak{u}_{n,1} := \mathfrak{u}_n \supset \mathfrak{u}_{n,2} \supset \cdots \supset \mathfrak{u}_{n,i} \supset \cdots \supset \mathfrak{u}_{n,n} \supset \mathfrak{u}_{n,n+1} := 0 \tag{1}$$

where $\mathfrak{u}_{n,i} := \sum_{j=i}^{n} P_{j-1}\partial_j$ for $i = 1, \ldots, n$. For all $i < j$,

$$[\mathfrak{u}_{n,i}, \mathfrak{u}_{n,j}] \subseteq \begin{cases} \mathfrak{u}_{n,i+1} & \text{if } i = j, \\ \mathfrak{u}_{n,j} & \text{if } i < j. \end{cases} \tag{2}$$

Proposition 2.1(2) of [3] states *that* (1) *is the derived series for the Lie algebra $\mathfrak{u}_n$, i.e., $(\mathfrak{u}_n)_{(i)} = \mathfrak{u}_{n,i+1}$ for all $i \geqslant 0$.*

1.2. *The group of automorphisms of the Lie algebra $\mathfrak{u}_n$*

In [4], the group of automorphisms G_n of the Lie algebra $\mathfrak{u}_n$ of triangular polynomial derivations is found ($n \geqslant 2$), it is isomorphic to an iterated semi-direct product [4, Theorem 5.3],

$$\mathbb{T}^n \ltimes \big(\mathrm{UAut}_K(P_n)_n \rtimes \big(\mathbb{F}'_n \times \mathbb{E}_n\big)\big)$$

where $\mathbb{T}^n$ is an algebraic n-dimensional torus, $\mathrm{UAut}_K(P_n)_n$ is an explicit factor group of the group $\mathrm{UAut}_K(P_n)$ of unitriangular polynomial automorphisms, $\mathbb{F}'_n$ and $\mathbb{E}_n$ are explicit groups that are isomorphic respectively to the groups $\mathbb{I}$ and $\mathbb{J}^{n-2}$ where $\mathbb{I} := (1 + t^2 K[[t]], \cdot) \simeq K^{\mathbb{N}}$ and $\mathbb{J} := (tK[[t]], +) \simeq K^{\mathbb{N}}$. It is shown that the *adjoint group* of automorphisms $\mathcal{A}(\mathfrak{u}_n)$ of the Lie algebra $\mathfrak{u}_n$ is equal to the group $\mathrm{UAut}_K(P_n)_n$ [4, Theorem 7.1]. Recall that the *adjoint group* $\mathcal{A}(\mathcal{G})$ of a Lie algebra $\mathcal{G}$ is generated by the elements $e^{\mathrm{ad}(g)} := \sum_{i \geqslant 0} \frac{\mathrm{ad}(g)^i}{i!} \in \mathrm{Aut}_K(\mathcal{G})$ where g runs through all the locally nilpotent elements of the Lie algebra $\mathcal{G}$ (an element g is a *locally nilpotent element* if the inner derivation $\mathrm{ad}(g) := [g, \cdot]$ of the Lie algebra $\mathcal{G}$ is a locally nilpotent derivation). The group G_n contains the semi-direct product $\mathbb{T}^n \ltimes \mathcal{T}_n$ where

$$\mathcal{T}_n := \big\{\sigma \in \mathrm{Aut}_K(P_n) \;\big|\; \sigma(x_1) = x_1,\ \sigma(x_i) = x_i + a_i \text{ where } a_i \in (x_1, \ldots, x_{i-1}),\ i = 2, \ldots, n\big\}$$

where $(x_1, \ldots, x_{i-1})$ is the maximal ideal of the polynomial algebra $P_{i-1} := K[x_1, \ldots, x_{i-1}]$ generated by the elements $x_1, \ldots, x_{i-1}$.

2. Proof of Theorem 1.1

Let $\varphi : \mathfrak{u}_n \to \mathfrak{u}_n$ be a monomorphism of the Lie algebra $\mathfrak{u}_n$. By Proposition 2.1(2) of [3], $(\mathfrak{u}_n)_{(i)} = \mathfrak{u}_{n,i+1}$ for all i. So, the descending chain of ideals (1) is the derived series for the Lie algebra $\mathfrak{u}_n$ of length $l(\mathfrak{u}_n) = n$ (by definition, this is the number of nonzero terms in the derived series). Clearly, $l(\mathfrak{u}_{n,2}) = n - 1$ and

$$l\big(\varphi(\mathfrak{u}_n)\big) = l(\mathfrak{u}_n) = n$$

($\varphi(\mathfrak{u}_n) \simeq \mathfrak{u}_n$). It follows that

$$\varphi(\mathfrak{u}_n) \nsubseteq \mathfrak{u}_{n,2}$$

since otherwise we would have $n = l(\varphi(\mathfrak{u}_n)) \leqslant l(\mathfrak{u}_{n,2}) = n - 1$, a contradiction. This means that $\partial_1' := \varphi(\partial_1) = \lambda_1 \partial_1 + u_1$ for some $\lambda_1 \in K^*$ and $u_i \in \mathfrak{u}_{n,2}$. We use induction on i to show that

$$\partial_i' := \varphi(\partial_i) = \lambda_i \partial_i + u_i, \quad i = 1, \ldots, n, \tag{3}$$

for some elements $\lambda_i \in K^*$ and $u_i \in \mathfrak{u}_{n,i+1}$. In particular, $\partial_n = \lambda_n \partial_n$. The initial step, $i = 1$, has already been established. Suppose that $i \geqslant 2$ and that (3) holds for all numbers $i' < i$. Since $\varphi((\mathfrak{u}_n)_{(j)}) \subseteq (\mathfrak{u}_n)_{(j)}$ for all $j \geqslant 1$, we have the inclusion $\varphi(\mathfrak{u}_{n,i}) = \varphi((\mathfrak{u}_n)_{(i-1)}) \subseteq (\mathfrak{u}_n)_{(i-1)} = \mathfrak{u}_{n,i}$ which implies that $\partial_i' = \lambda_i \partial_i + u_i$ for some elements $\lambda_i \in P_{i-1}$ and $u_i \in \mathfrak{u}_{n,i+1}$. It remains to show that $\lambda_i \in K^*$. This fact follows from the commutation relations $[\partial_j', \partial_i'] = 0$ for $j = 1, \ldots, i-1$ ($0 = \varphi([\partial_j, \partial_i]) = [\partial_j', \partial_i']$). In more detail, for $j = i - 1$,

$$0 = \big[\partial_{i-1}', \partial_i'\big] = [\lambda_{i-1}\partial_{i-1} + u_{i-1}, \lambda_i \partial_i + u_i] = \lambda_{i-1}\partial_{i-1}(\lambda_i)\partial_i + v_{i-1}$$

for some element $v_{i-1} \in \mathfrak{u}_{n,i+1}$. Therefore, $\partial_{i-1}(\lambda_i) = 0$, i.e., $\lambda_i \in P_{i-2}$. Now, we use a second downward induction on j starting on $j = i - 1$ to show that

$$\lambda_i \in P_j \quad \text{for all } j = 1, \ldots, i-1. \tag{4}$$

The initial step, $j = i - 1$, has been just proved. Suppose that (4) is true for all $j = k, \ldots, i-1$. In particular, $\lambda_i \in P_k = K[x_1, \ldots, x_k]$. We have to show that $\lambda_i \in P_{k-1}$. For, we use the equality $[\partial_k', \partial_i'] = 0$:

$$0 = [\lambda_k \partial_k + u_k, \lambda_i \partial_i + u_i] = \lambda_k \partial_k(\lambda_i)\partial_i + v_k$$

for some element $v_k \in \mathfrak{u}_{n,i+1}$ ($[u_k, \lambda_i \partial_i] \in \mathfrak{u}_{n,i+1}$ since $\lambda_i \in P_k$ and $[\bigoplus_{k+1 \leqslant j \leqslant i} P_{j-1}\partial_j, \lambda_i \partial_i] = 0$). Therefore, $\partial_k(\lambda_i) = 0$, i.e., $\lambda_i \in P_{k-1}$. By induction on j, (4) holds. In particular, for $j = 1$: $\lambda_i \in P_{1-1} = P_0 = K$. We have to show that $\lambda_i \neq 0$. Notice that $\mathfrak{u}_{n,i} = \bigoplus_{j=i}^n P_{j-1}\partial_j$, $l(\mathfrak{u}_{n,i}) = n - i + 1$ and $\varphi(\mathfrak{u}_{n,i}) \subseteq \mathfrak{u}_{n,i}$. The monomorphism φ respects the Lie subalgebra $\mathcal{G} = K\partial_i + \mathfrak{u}_{n,i+1}$ of the Lie algebra $\mathfrak{u}_n$, i.e., $\varphi(\mathcal{G}) \subseteq \mathcal{G}$. The inclusion of Lie algebras $\mathcal{G} \subseteq \mathfrak{u}_{n,i}$ yields the inequality $l(\mathcal{G}) \leqslant l(\mathfrak{u}_{n,i}) = n - i + 1$ (the equality follows from the fact that $(\mathfrak{u}_n)_{(j)} = \mathfrak{u}_{n,j+1}$ for all $j \geqslant 0$). The vector space

$$\mathcal{H} = K\partial_i + K[x_i]\partial_{i+1} + K[x_i, x_{i+1}]\partial_{i+2} + \cdots + K[x_i, \ldots, x_{n-1}]\partial_n$$

is a Lie subalgebra of $\mathcal{G}$ which is isomorphic to the Lie algebra $\mathfrak{u}_{n-i+1}$. Therefore, $l(\mathcal{H}) = l(\mathfrak{u}_{n-i+1}) = n - i + 1$. The inclusion of Lie algebra $\mathcal{H} \subseteq \mathcal{G}$ yields the inequality $n - i + 1 = l(\mathcal{H}) \leqslant l(\mathcal{G})$. Therefore, $l(\mathcal{G}) = n - i + 1$.

Suppose that $\lambda_i = 0$, we seek a contradiction. In that case, $\varphi(\mathcal{G}) \subseteq \mathfrak{u}_{n,i+1}$ and so

$$n - i + 1 = l(\mathcal{G}) = l\big(\varphi(\mathcal{G})\big) \leqslant l(\mathfrak{u}_{n,i+1}) = n - i,$$

a contradiction.

Therefore, (3) holds. By Theorem 3.6(2) of [4], there exists a unique automorphism $\sigma \in \mathbb{T}^n \ltimes \mathcal{T}_n \subseteq G_n$ such that $\sigma(\partial_i) = \partial_i'$ for $i = 1, \ldots, n$. By replacing the monomorphism φ by the monomorphism $\sigma^{-1}\varphi$, without loss of generality we can assume that

$$\partial_i' = \partial_i \quad \text{for all } i = 1, \ldots, n.$$

The vector space $\mathfrak{u}_n$ is the union $\bigcup_{i \geqslant 0} N_i$ of vector subspaces $N_i := \{u \in \mathfrak{u}_n \mid \delta_j^{i+1}(u) = 0,\ j = 1, \ldots, n-1\}$ where $\delta_j = \mathrm{ad}(\partial_j)$. Clearly, $N_i = \bigoplus_{j=1}^n N_i \cap P_{j-1}\partial_j$ and $N_i \cap P_{j-1}\partial_j = \bigoplus\{Kx^\alpha \mid \alpha = (\alpha_1, \ldots, \alpha_{j-1}) \in \mathbb{N}^{j-1},\ \alpha_k \leqslant i \text{ for } k = 1, \ldots, j-1\}$. In particular,

$$\dim_K(N_i) < \infty \quad \text{for all } i \geqslant 0.$$

By the very definition of the vector spaces N_i and the fact that $\varphi(\partial_i) = \partial_i$ for $i = 1, \ldots, n-1$,

$$\varphi(N_i) \subseteq N_i \quad \text{for all } i \geqslant 0.$$

Since the linear map φ is an injection and the vector spaces N_i are finite dimensional, we have $\varphi(N_i) = N_i$ for all $i \geqslant 0$, i.e., φ is a bijection.

Acknowledgements

The work is partly supported by the Royal Society and EPSRC.

References

[1] V.V. Bavula, The Jacobian Conjecture$_{2n}$ implies the Dixmier Problem$_n$, arXiv:math.RA/0512250.
[2] V.V. Bavula, An analogue of the Conjecture of Dixmier is true for the algebra of polynomial integro-differential operators, arXiv:1011.3009 [math.RA].
[3] V.V. Bavula, Lie algebras of unitriangular polynomial derivations and an isomorphism criterion for their Lie factor algebras, arXiv:1204.4908 [math.RA].
[4] V.V. Bavula, The groups of automorphisms of the Lie algebras of unitriangular polynomial derivations, arXiv:1204.4910 [math.AG].
[5] A. Belov-Kanel, M. Kontsevich, The Jacobian Conjecture is stably equivalent to the Dixmier Conjecture, Mosc. Math. J. 7 (2) (2007) 209–218, arXiv:math.RA/0512171.
[6] J. Dixmier, Sur les algèbres de Weyl, Bull. Soc. Math. France 96 (1968) 209–242.
[7] Y. Tsuchimoto, Endomorphisms of Weyl algebra and p-curvatures, Osaka J. Math. 42 (2) (2005) 435–452.