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Harmonic Analysis

Revisiting Landau's density theorems for Paley–Wiener spaces ☆

Retour sur les théorèmes de densité de Landau dans les espaces de Paley–Wiener

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A B S T R A C T

We present a surprisingly simple approach to Landau's density theorems for sampling and interpolation, which provides stronger versions of these results. In particular, we extend the interpolation theorem to unbounded spectra.

R É S U M É

On présente ici une approche simple, et plutôt surprenante, des théorèmes de densité de Landau fournissant, pour l'échantillonnage et l'interpolation, des versions plus fortes des résultats connus. En particulier, on étend le théorème d'interpolation au spectre non borné.

1. Introduction

Let S be a set of finite measure in $\mathbb{R}$. The Paley–Wiener space PW_S is defined as the subspace in $L^2(\mathbb{R})$, consisting of all functions of the form

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_S F(t) e^{itx}\, dt, \tag{1}$$

where $F \in L^2(S)$.

We assume below that Λ is a uniformly discrete (u.d.) subset of $\mathbb{R}$, i.e. for some $\delta > 0$,

$$\inf_{\lambda \neq \lambda'\ \lambda,\lambda' \in \Lambda} \left|\lambda - \lambda'\right| > \delta.$$

A set $\Lambda \subseteq \mathbb{R}$ is called a (stable) sampling set (SS) for PW_S if there exists a constant $C > 0$ such that

$$\|f\|^2_{L_2(\mathbb{R})} \leqslant C \sum_{\lambda \in \Lambda} \left|f(\lambda)\right|^2 \quad \forall f \in PW_S,$$

and an interpolation set (IS) if for every vector $c(\lambda) \in l^2(\Lambda)$ there is $f \in PW_S$ with $f(\lambda) = c(\lambda)$ for all $\lambda \in \Lambda$.

In the classical case, when the spectrum S is an interval, SS and IS were essentially characterized by A. Beurling [1] and J.-P. Kahane [3] in terms of the uniform densities $D^-(\Lambda)$ and $D^+(\Lambda)$ (for the definition of these densities see Sections 3 and 4). The necessity part in these results was extended by H. Landau [4] to the general case.

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Theorem A *(Landau). Let S be a bounded measurable set on $\mathbb{R}$ and Λ be a u.d. set on (another copy) of $\mathbb{R}$.*

(i) *If Λ is a SS for PW_S then*

$$D^-(\Lambda) \geqslant \frac{|S|}{2\pi}. \tag{2}$$

(ii) *If Λ is an IS for PW_S then*

$$D^+(\Lambda) \leqslant \frac{|S|}{2\pi}. \tag{3}$$

The proof in [4] is based on a delicate analysis of eigen-values of products of certain projectors in the time and frequency domains.

In the case, when S is a finite union of intervals, a simpler proof of Theorem A has been given in [2], based on the approach developed in [7].

Observe that while Landau's sampling theorem is automatically extendable to the case of unbounded spectra, the corresponding problem for interpolation apparently remained open. Keeping in mind that functions with unbounded spectrum are not necessary analytic, one may expect that the restrictions for interpolation by such functions will not be as strong as in condition (3). This is indeed so for Bernstein spaces B_S of bounded continuous functions with spectrum in S. It was proved in [5,6] that such spaces may admit IS with large density, compared to the measure of S. It turns out that for Paley–Wiener spaces the situation is different.

In this Note we give a new and simple approach to the subject which allows one to avoid Landau's spectral technique and put the results in a more general context. In particular we prove that condition (3) is still necessary for IS in Paley–Wiener spaces with unbounded spectra.

Notations. We denote the cardinality of a finite set A by $|A|$ and the measure of a set $S \subseteq \mathbb{R}$ by $|S|$. We use the notation $\langle -, - \rangle$ for the scalar product in the (complex) space $L^2(S)$. In addition, we denote by C constants which may change from line to line.

2. Bi-orthogonal systems in Paley–Wiener spaces

Our key observation follows:

Lemma 1. *Let $S \subset \mathbb{R}$ be a set of finite measure, V – a closed subspace of PW_S, $\{f_k\}$ and $\{g_k\}$ dual Riesz bases of V. Then,*

$$0 \leqslant \sum f_k(x)\overline{g_k(x)} \leqslant \frac{|S|}{2\pi}, \quad \forall x \in \mathbb{R}. \tag{4}$$

Moreover, if $V = PW_S$ then the right inequality in (4) becomes an equality.

Proof. For every $x \in \mathbb{R}$ consider the harmonic $e_x(t) := (2\pi)^{-1/2}e^{-ixt}$ as an element of $L^2(S)$. Let F_k correspond to f_k by (1), so $f_k(x) = \langle F_k, e_x \rangle$. Similarly let G_k correspond to g_k. Then $\{F_k\}$ and $\{G_k\}$ are dual Riesz bases in the subspace $\hat{V} \subset L^2(S)$, which is the Fourier image of V.

We have: $A(x) := \sum f_k(x)\overline{g_k(x)} = \sum \langle F_k, e_x \rangle \langle e_x, G_k \rangle = \langle \sum \langle e_x, G_k \rangle F_k, e_x \rangle$. Clearly, $\sum \langle e_x, G_k \rangle F_k = Pe_x$, where P is the orthogonal projection of $L^2(S)$ on $\hat{V}$. So $A(x) = \langle Pe_x, e_x \rangle = \|P(e_x)\|^2$, and (4) follows. □

Lemma 1 is ready for applications to the interpolation problem. For sampling we need it in a slightly different form.

Let $\{u_k\}$ be a (normalized) frame in a Hilbert space H. It does not admit a bi-orthogonal partner unless it is a Riesz basis. However, there is another (the dual) frame $\{v_k\}$ in H such that:

$$\left|\langle u_k, v_k \rangle\right| \leqslant 1, \quad \forall k, \tag{5}$$

and every vector $f \in H$ has a decomposition as $f = \sum_k \langle f, v_k \rangle u_k$. (See, for example, [8].)

Lemma 2. *Let f_k be a frame in PW_S and g_k be its dual frame. Then*

$$\sum f_k(x)\overline{g_k(x)} = \frac{|S|}{2\pi}. \tag{6}$$

The proof is the same as above.

3. Interpolation

The upper uniform density is defined as $D^+(\Lambda) := \lim_{r\to\infty} \frac{\max_{x\in\mathbb{R}} |\Lambda \cap [x,x+r]|}{r}$.

Theorem 1. *Let $S \subset \mathbb{R}$ be an arbitrary (not necessary bounded) set of finite measure. If Λ is an IS for PW_S then condition (3) holds.*

In fact we prove a stronger result which requires only the interpolation of delta functions. Recall the following

Definition 1. A (normalized) system of vectors $\{u_n\}$, in a Hilbert space H is called *uniformly minimal*, if the distance from each of these vectors to the span of all others is bigger than a positive constant.

Theorem 2. *Let $h \in PW_S$ and Λ be a u.d. set. If the family of translates $\{h_\lambda\}$, $h_\lambda(x) := h(x-\lambda)$, is uniformly minimal in PW_S then inequality (3) holds.*

Taking $h = \hat{\mathbb{1}}_S$ we get

Theorem 3. *If the exponential system $E(\Lambda) := \{e^{i\lambda t}, \lambda \in \Lambda\}$ is uniformly minimal in $L^2(S)$ then (3) holds.*

For compact spectrum this was proved in [5].

Notice that the condition in Definition 1 is equivalent to the existence of a bounded system $\{v_n\}$ in H such that $\{u_n\}$ and $\{v_n\}$ are bi-orthogonal. If Λ is an IS for PW_S then the exponential system $E(\Lambda)$ is uniformly minimal in $L^2(S)$, see [5]. So Theorem 3 implies Theorem 1.

Proof of Theorem 2. Fix $\epsilon > 0$ and choose $b > 0$ for which

$$\int\limits_{|x|>b} \left|h(x)\right|^2 \mathrm{d}x < \epsilon^2. \tag{7}$$

Denote by $\{g_\lambda\}_{\lambda\in\Lambda}$ a dual system of $\{h_\lambda\}$ which satisfies $\|g_\lambda\| < C$ for all $\lambda \in \Lambda$.

Given any interval $I \subset \mathbb{R}$ consider the space $V = span\{h_\lambda, \lambda \in \Lambda \cap I\}$ and denote by P_V the orthogonal projection from PW_S to this space. The systems $\{h_\lambda\}_{\lambda\in\Lambda\cap I}$ and $\{P_V g_\lambda\}_{\lambda\in\Lambda\cap I}$ are dual Riesz bases in the finite-dimensional space V. Hence, we have from (4):

$$\sum_{\lambda\in\Lambda\cap I} h(x-\lambda)\overline{P_V g_\lambda(x)} \leqslant \frac{|S|}{2\pi}.$$

Set $|I| = r$ and let I_b be an interval concentric with I of length $r+2b$. Integrating over this interval we get

$$\int\limits_{I_b} \sum_{\lambda\in\Lambda\cap I} h(x-\lambda)\overline{P_V g_\lambda(x)}\,\mathrm{d}x \leqslant \frac{|S|(r+2b)}{2\pi}. \tag{8}$$

On the other hand, for any $\lambda \in \Lambda \cap I$ we have,

$$\int\limits_{I_b} h(x-\lambda)\overline{P_V g_\lambda(x)}\,\mathrm{d}x = 1 - \int\limits_{\mathbb{R}\setminus I_b} h(x-\lambda)\overline{P_V g_\lambda(x)}\,\mathrm{d}x.$$

We use (7) to estimate the last integral,

$$\left|\int\limits_{\mathbb{R}\setminus I_b} h(x-\lambda)\overline{P_V g_\lambda(x)}\,\mathrm{d}x\right|^2 \leqslant C \int\limits_{\mathbb{R}\setminus I_b} \left|h(x-\lambda)\right|^2 \mathrm{d}x \leqslant C \int\limits_{|x|>b} \left|h(x)\right|^2 \mathrm{d}x < C\epsilon^2.$$

Combining this estimate with (8) we find that

$$(1 - C\epsilon)\frac{\max_{x\in\mathbb{R}} |\Lambda \cap [x, x+r]|}{r} \leqslant \frac{|S|(r+2b)}{2\pi r}.$$

Letting r tend to infinity, and recalling that $\epsilon > 0$ was arbitrary, we obtain inequality (3). $\square$

4. Sampling

Here we discuss the sampling problem or, in geometric language, exponential frames in $L^2(S)$.

The lower uniform density is defined as: $D^-(\Lambda) := \lim_{r\to\infty} \frac{\min_{x\in\mathbb{R}} |\Lambda\cap[x,x+r]|}{r}$.

Theorem 4. *Let S be a bounded set, Λ be as above and $h \in PW_S$. Suppose there are functions $\{g_\lambda\}_{\lambda\in\Lambda}$ in PW_S with bounded norms such that:*

(i) $\sum_{\lambda\in\Lambda} |g_\lambda(x)|^2 < C, \ \forall x \in \mathbb{R}$;
(ii) $|\int_{\mathbb{R}} h(x-\lambda)\overline{g_\lambda(x)}\,\mathrm{d}x| \leqslant 1, \ \forall \lambda \in \Lambda$.

Then

$$D^-(\Lambda) \geqslant \lim_{r\to\infty} \inf_{|I|=r} \frac{1}{|I|}\left|\int\limits_I A(x)\,\mathrm{d}x\right| \tag{9}$$

where $A(x) := \sum_{\lambda\in\Lambda} h(x-\lambda)\overline{g_\lambda(x)}$.

Proof. Clearly the right part of (9) is finite, denote it by α. Fix $\epsilon > 0$. One can easily check that, for $b > 0$ large enough, the following inequality holds uniformly with respect to x:

$$\sum_{|x-\lambda|>b} |h(x-\lambda)|^2 < \epsilon^2. \tag{10}$$

Fix b satisfying both conditions (7) and (10).

Let $I \subset \mathbb{R}$ be a sufficiently large interval. Denote by I_b and I^b concentric intervals of length $|I| + 2b$ an $|I| - 2b$, respectively.

We have $|\int_I A(x)\,\mathrm{d}x| > (\alpha - \epsilon)|I|$. Decompose $A = A_1 + A_2 + A_3$, where in each sum we include members of Λ belonging to I^b, $R \setminus I_b$ and $I_b - I^b$ respectively.

Now, condition (10) implies the simple estimates: $\int_I |A_2| < C|I|\epsilon$ and $\int_I |A_3| < C\frac{b}{\delta}$ (here δ is the separation constant for Λ).

Finally, for each member in A_1, condition (7) and Cauchy–Schwartz inequality imply:

$$\left|\int\limits_I h(x-\lambda)\overline{g_\lambda(x)}\,\mathrm{d}x\right| \leqslant \left|\int\limits_{\mathbb{R}} h(x-\lambda)\overline{g_\lambda(x)}\,\mathrm{d}x\right| + \left|\int\limits_{\mathbb{R}\setminus I} h(x-\lambda)\overline{g_\lambda(x)}\,\mathrm{d}x\right| < 1 + C\epsilon.$$

Summing up we get: $(\alpha - \epsilon)|I| < (1 + C\epsilon)|\Lambda \cap I| + C\epsilon|I| + C_1(\epsilon)$, which implies (9). $\square$

Corollary 1. *If the family of translates* $\{h(x-\lambda)\}$ *is a frame in* PW_S *then condition* (2) *holds.*

Proof. Denote by $\{g_\lambda\}$ the dual frame of $\{h(x-\lambda)\}$. Let $G_\lambda \in L^2(S)$ be the functions corresponding to g_λ according to (1). In particular, these functions satisfy

$$\sum |\langle F, G_\lambda\rangle|^2 \leqslant C\|F\|^2, \quad \forall F \in L^2(S).$$

Putting $F(t) = e^{-ixt}$ we find that condition (i) of Theorem 4 is satisfied. Due to (5) we have condition (ii). Now (9) and (6) give (2). $\square$

Taking $h = \hat{\mathbb{1}}_S$ in Corollary 1, we get part (i) of Theorem A. Notice that Theorem 4 may be useful not only for frames.

5. Remarks

1. In the case when S is a finite union of intervals, local estimates for $|\Lambda \cap I|$, with the logarithmic second term, are true (Landau [4]). For example: if Λ is an IS for PW_S and I is an interval, $|I| > 2$, then

$$|\Lambda \cap I| \leqslant \frac{|S|}{2\pi}|I| + C(\Lambda, S)\log|I|.$$

   These estimates also can be proved by the technique above.
2. One can verify that the results stated here can be extended to higher dimensions (where the corresponding densities are defined as in [4]).

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