C. R. Acad. Sci. Paris, Ser. I 350 (2012) 393–397

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Harmonic Analysis

Refined inequalities on graded Lie groups

Inégalités précisées sur les groupes de Lie gradués

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ARTICLE INFO

Article history:
Received 12 March 2012
Accepted after revision 17 April 2012
Available online 5 May 2012

Presented by Jean-Michel Bony

ABSTRACT

We construct a Littlewood–Paley decomposition associated to a Rockland operator on graded Lie groups, which allows us to deduce refined Gagliardo–Nirenberg, Sobolev and Hardy inequalities.

RÉSUMÉ

On construit une décomposition de Littlewood–Paley associée à un opérateur de Rockland sur les groupes de Lie gradués, qui permet de déduire des inégalités précisées de type Gagliardo–Nirenberg, Sobolev et Hardy.

Version française abrégée

L'objectif de cette Note est de présenter une extension au cas des groupes de Lie gradués d'inégalités (précisées) connues dans le cas euclidien (Sobolev, Gagliardo–Nirenberg, Hardy) où les espaces fonctionnels sont construits à partir d'un opérateur de Rockland. Le résultat principal consiste à utiliser cet opérateur pour construire une décomposition de type Littlewood–Paley sur ce groupe, laquelle conserve beaucoup des propriétés du cas euclidien (notamment vis-à-vis des dilatations). On considère donc un groupe de Lie gradué G muni de la mesure de Haar μ, de dimension homogène Q, ainsi qu'un opérateur de Rockland positif L de degré D. On sait depuis [12] que si a est une fonction de $\mathcal{C}_0^\infty(\mathbb{R})$, alors il existe $h_a \in \mathcal{S}(G)$ tel que $\forall f \in \mathcal{S}(G), a(L)f = f * h_a$. Il est démontré dans cette note le théorème suivant :

Théorème 0.1. *Soit $a \in \mathcal{C}_0^\infty(\mathbb{R})$ et h_a vérifiant $a(L)f = f * h_a$. Alors pour tout $t > 0$ et tout $f \in \mathcal{S}(G)$ on peut écrire $a(t^{-D}L)f = f * h_a^{(t)}$ avec $h_a^{(t)} := t^Q \delta_t h_a$. En outre si $a(0) = 0$, alors h_a est de moyenne nulle. Enfin il existe une fonction intégrable décroissante $H_a : \mathbb{R}_+ \to \mathbb{R}_+$ telle que pour tout $x \in G$, on ait $|h_a(x)| \leqslant H_a(\rho(x))$.*

Ce résultat permet de construire un découpage dyadique se basant sur la décomposition spectrale de L : on considère ψ et ϕ des fonctions régulières supportées respectivement sur la boule et une couronne unité, telles que pour tout $t \in \mathbb{R}$, $1 = \psi(t) + \sum_{j\in\mathbb{N}} \phi(2^{-j}t)$ et pour tout $t \neq 0$, $1 = \sum_{j\in\mathbb{Z}} \phi(2^{-j}t)$; alors on a la décomposition $\mathrm{Id} = \psi(L) + \sum_{j\in\mathbb{N}} \phi(2^{-jD}L) = \sum_{j\in\mathbb{Z}} \phi(2^{-jD}L)$. Le théorème précédent indique que si $\Delta_j := \phi(2^{-jD}L)$ et $S_j := \psi(2^{-jD}L)$ (et on définit de même les

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1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.04.014

opérateurs homogènes $\dot{\Delta}_j$ et $\dot{S}_j$) alors $\forall f \in \mathcal{S}(G)$, $\Delta_j f = f * \Phi_j$ et $S_j f = f * \Psi_j$ avec $\Phi_j = 2^{jQ}\delta_{2^j}\Phi_0$ et $\Psi_j = 2^{jQ}\delta_{2^j}\Psi_0$ avec Φ_0 et Ψ_0 dans $\mathcal{S}(G)$ et Φ_0 de moyenne nulle. Cela permet de définir des espaces de Besov, et la définition des espaces de Sobolev via ces opérateurs coïncide avec $\|f\|_{H^s(G)} := \|(1+L)^{\frac{s}{D}} f(x)\|_{L^2}$. Des inégalités de Bernstein découlent de ces propriétés (voir le lemme 2.2) et enfin on peut reprendre les démonstrations des inégalités précisées du cas euclidien (ou Heisenberg) développées dans [3–5] pour obtenir les résultats suivants (voir les corollaires 3.1, 3.2, 3.3 et 3.4 pour des énoncés précis) :

Sobolev précisée :

$$\|f\|_{L^p(G)} \leqslant C(p-q)^{\frac{1}{p}} \|f\|_{\dot{B}^s_{q,q}(G)}^{1-\frac{qs}{Q}} \|f\|_{\dot{B}^{s-\frac{Q}{q}}_{\infty,\infty}(G)}^{\frac{qs}{Q}}, \quad \frac{1}{p} = \frac{1}{q} - \frac{s}{Q}.$$

Moser–Trudinger :

$$\int_G \left(\exp\left\{c\left(\frac{|f(x)|}{\|f\|_{H^{\frac{Q}{2}}}}\right)^2\right\} - 1\right) \mathrm{d}\mu(x) \leqslant C.$$

Gagliardo–Nirenberg :

$$\|f\|_{\dot{W}^\sigma_p(G)} \leqslant C\|f\|^\theta_{L^q(G)} \|f\|^{1-\theta}_{\dot{W}^s_r(G)}, \quad \frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r},\ \theta = 1 - \frac{\sigma}{s}.$$

Hardy précisée :

$$\left(\int_G \frac{|f(x)|^q}{\rho(x)^{sq}} \mathrm{d}\mu(x)\right)^{\frac{1}{q}} \leqslant C\|f\|_{\dot{B}^s_{q,q}(G)}^{1-\frac{qs}{Q}} \|f\|_{\dot{B}^{s-\frac{Q}{q}}_{\infty,q}(G)}^{\frac{qs}{Q}}.$$

1. Introduction

The aim of this Note is to extend to graded Lie groups well-known (in the $\mathbb{R}^d$ case) inequalities such as refined Gagliardo–Nirenberg, Sobolev and Hardy inequalities. Their main feature is that they are invariant under oscillations, fractal transforms and "chirps" (see [3] and [4] for the sharpness of these inequalities). They are classically proved by Littlewood–Paley theory, though other proofs are also known in some contexts, see for instance [8]. In [3] and [5], such inequalities were extended to the Heisenberg group thanks to the Littlewood–Paley decomposition. To generalize them to graded Lie groups, our main task in this Note is to construct an adequate Littlewood–Paley decomposition on graded Lie groups.

1.1. Graded Lie groups

Let us recall basic facts about graded Lie groups (see [9] and the references therein). A simply connected nilpotent Lie group G is graded if its left-invariant Lie algebra $\mathfrak{g}$ (assumed real-valued and of finite dimension n) is endowed with a vector space decomposition $\mathfrak{g} = \bigoplus_{1 \leqslant k \leqslant \infty} \mathfrak{g}_k$ (where all but finitely many of the $\mathfrak{g}_k$'s are $\{0\}$) such that $[\mathfrak{g}_k, \mathfrak{g}_{k'}] \subset \mathfrak{g}_{k+k'}$. Via the exponential map one identifies G and $\mathfrak{g}$. Then the group law on G is a polynomial map. The group G is endowed with a smooth left-invariant measure $\mu(x)$ called the Haar measure. There is a natural family of dilations on $\mathfrak{g}$ defined for $t > 0$ as follows: if X belongs to $\mathfrak{g}$, we can write $X = \sum X_k$ with $X_k \in \mathfrak{g}_k$, and then $\delta_t X := \sum t^k X_k$. The dilation δ_t on G is defined via the exponential map. The Jacobian of the dilation δ_t is t^Q, where $Q \in \mathbb{N}$ is the homogeneous dimension of G. The action of the left-invariant vector fields $X_k \in \mathfrak{g}_k$ changes the homogeneity by $X_k(f \circ \delta_t) = t^k X_k(f) \circ \delta_t$; in what follows, we shall write $\delta_t f := f \circ \delta_t$. Finally it is convenient to use a norm ρ which is homogeneous in the sense that it respects dilations: $x \mapsto \rho(x)$ satisfies $\forall x \in G$, $\rho(x^{-1}) = \rho(x)$, $\rho(\delta_t x) = t\rho(x)$ and $\rho(x) = 0$ iff $x = 0$. Then the distance $d(x, y)$ between two points x and y of G is defined obviously by $d(x, y) = \rho(x^{-1} \cdot y)$.

1.2. Function spaces on G

The Schwartz space $\mathcal{S}(G)$ is defined as the set of smooth functions on G such that for all α in $\mathbb{N}^d$, for all p in $\mathbb{N}$, $x \mapsto \rho(x)^p \mathcal{X}^\alpha f(x) \in L^\infty(G)$, where $\mathcal{X}^\alpha$ denotes a product of $|\alpha|$ left-invariant vector fields. As usual its topological dual $\mathcal{S}'(G)$ is the space of tempered distributions. The Schwartz space $\mathcal{S}(G)$ is dense in the Lebesgue spaces $L^p(G; \mu)$ for $p \in [1, \infty[$. One can also define, as in the Euclidean case, Lorentz spaces $L^{p,q}(G)$. We refer to [4,7,13,14] for details. Finally we recall that the convolution of two functions f and g on G is given by

$$f * g(x) = \int_G f\left(x \cdot y^{-1}\right) g(y)\, \mathrm{d}\mu(y) = \int_G f(y) g\left(y^{-1} \cdot x\right) \mathrm{d}\mu(y)$$

and the usual Young inequalities are valid (in L^p and $L^{p,q}$).

We consider a positive Rockland operator L, that is a positive, self-adjoint, differential left-invariant operator L on G which is homogeneous of degree D with respect to the dilations: $\delta_{t^{-1}} L \delta_t = t^D L$, and for which the pull-back by π of the operator L is injective on the space of $\mathcal{C}^\infty$ vectors for every nontrivial irreducible unitary representation π of G. Compared to the sub-Laplacian $\Delta_G := \sum_{j=1}^n X_j^2$, a Rockland operator on the one hand is in some sense less derivative-consuming. On the other hand, it takes fully into account the homogeneity of the group.

For $s \geqslant 0$, we define the Sobolev space $H^s(G)$ on G as the set of f such that $(1+L)^{\frac{s}{D}} f$ belongs to $L^2(G)$. The norm on $H^s(G)$ is given by $\|f\|_{H^s(G)} := \|(1+L)^{\frac{s}{D}} f\|_{L^2}$. Like in the Euclidean space, the Sobolev spaces $H^{-s}(G)$ are defined by duality and homogeneous Sobolev spaces may be defined by the (semi)-norm $\|f\|_{\dot H^s(G)} := \|L^{\frac{s}{D}} f\|_{L^2}$. More generally, for $\sigma < \frac{Q}{p}$ we define the Sobolev spaces $\dot W_p^\sigma(G)$ by $\|f\|_{\dot W_p^\sigma(G)} := \|L^{\frac{\sigma}{D}} f\|_{L^p(G)}$.

2. Littlewood–Paley theory

The dyadic partition of unity we are going to define is based on a positive Rockland operator L. This construction is similar to that carried out in [10], the difference lies first in the fact that one considers a Rockland operator and not the sub-Laplacian, and moreover that one can take advantage of the graded structure of G to infer additional properties: Theorem 2.1 provides the main new result, on the kernel of $a(L)$ for any $a \in \mathcal{C}_0^\infty(\mathbb{R})$. We consider a spectral resolution $\mathrm{d}E(\lambda)$ of the Rockland operator L on $L^2(G)$, implying that for a continuous and bounded function a on $\mathbb{R}$, $a(L)f := \int_0^\infty a(\lambda)\,\mathrm{d}E(\lambda)\, f$, by standard functional calculus for self-adjoint operators. According to Hulanicki's theorem [12], if a belongs to $\mathcal{C}_0^\infty(\mathbb{R})$, there is $h_a \in \mathcal{S}(G)$ such that for all $f \in \mathcal{S}(G)$, $a(L)f = f * h_a$.

Theorem 2.1. *Let $a \in \mathcal{C}_0^\infty(\mathbb{R})$ and h_a be such that $a(L)f = f * h_a$. Then*

$$\forall t > 0,\ \forall f \in \mathcal{S}(G), \quad a\big(t^{-D}L\big)f = f * h_a^{(t)} \quad \text{with } h_a^{(t)} := t^Q \delta_t h_a. \tag{1}$$

Furthermore, if $a(0) = 0$, then h_a has vanishing mean. Finally there is a nonincreasing, integrable function $H_a : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\forall x \in G, \quad \big|h_a(x)\big| \leqslant H_a\big(\rho(x)\big). \tag{2}$$

Proof. Let us first prove (1). Let $t > 0$ be given and denote $\delta_t f := f \circ \delta_t$. By homogeneity of L, we have $\delta_t \circ L \circ \delta_t^{-1} = t^{-D} L$. Of course L^n is homogeneous of degree nD, so for any polynomial function P, we have $\delta_t \circ P(L) \circ \delta_t^{-1} = P(t^{-D}L)$. This implies that for all $a \in \mathcal{C}_0^\infty(\mathbb{R})$, $\delta_t \circ a(L) \circ \delta_t^{-1} = a(t^{-D}L)$. Therefore, for f in $\mathcal{S}(G)$, we have $a(t^{-D}L)f = \delta_t \circ a(L) \circ \delta_t^{-1} f = \delta_t((\delta_t^{-1} f) * h_a)$. Observing that for all $f, g \in \mathcal{S}(G)$, $f * \delta_t g = t^{-Q} \delta_t((\delta_t^{-1} f) * g)$, (1) is proved. Now let us assume $a(0) = 0$. Define $\phi_1(\lambda) := \lambda^{-1} a(\lambda) \in \mathcal{C}_0^\infty(\mathbb{R})$. Then $a(L)f = L\phi_1(L)f$ which gives $h_a = L h_{\phi_1}$. We infer that

$$\int_G h_a(x)\,\mathrm{d}\mu(x) = \int_G L h_{\phi_1}(x)\,\mathrm{d}\mu(x) = 0.$$

Finally let us prove the bound (2). We recall that by [12], there is a function k_t such that $\forall t > 0$, $\forall f \in \mathcal{S}(G)$, one has $e^{-tL} f = f * k_t$, and there exists a holomorphic extension of k_t in $\{z \in \mathbb{C},\ \mathrm{Re}(z) > 0\}$ such that $k_z \in \mathcal{S}(G)$. Now let us write

$$a(L) = e^{-L} \int_0^\infty e^\lambda a(\lambda)\,\mathrm{d}E(\lambda) = (2\pi)^{-1} e^{-L} \int_0^\infty \int_{-\infty}^\infty \widehat{\theta}(\xi) e^{i\xi\lambda}\,\mathrm{d}E(\lambda)\,\mathrm{d}\xi$$

where $\theta(\lambda) = e^\lambda a(\lambda)$ and $\widehat{\theta}$ denotes the Fourier transform of θ, whence

$$a(L)f = f * (2\pi)^{-1} \int_{-\infty}^\infty \widehat{\theta}(\xi) k_{1-i\xi}\,\mathrm{d}\xi.$$

Therefore, the function h_a is given by $h_a := (2\pi)^{-1} \int_{-\infty}^\infty \widehat{\theta}(\xi) k_{1-i\xi}\,\mathrm{d}\xi$. But it is proved in [1] for instance that the holomorphic extension of the heat kernel k_t satisfies Gaussian bounds in $\{z, |\arg z| < \pi/2 - \varepsilon\}$ for all $\varepsilon > 0$, of the type

$$\big|k_z(x)\big| \leqslant \frac{C_1}{|z|^{\frac{Q}{D}}} \exp\left(-C_2^{-1} \left(\frac{\rho^D(x)}{|z|}\right)^{\frac{1}{D-1}}\right).$$

Moreover an examination of the proof of [1] shows that the constants C_1, C_2 blow up like powers of ε. Using that estimate with $z = 1 - i\xi$ (hence $\varepsilon \sim |\xi|^{-1}$) along with the fact that $\widehat{\theta}$ belongs to $\mathcal{S}(\mathbb{R})$, ends the proof of the theorem. □

Note that (2) gives a control of h_a by a function of the norm of x which is useful for proving Gagliardo–Nirenberg inequalities below.

We now come to the definition of Littlewood–Paley operators on the group G. We denote by $\mathcal{B}$ the unit ball of $\mathbb{R}$ and $\mathcal{C}$ a unit ring: $\mathcal{B} = \{t \in \mathbb{R},\ |t| \leqslant 1\}$, $\mathcal{C} = \{t \in \mathbb{R},\ 1/2 \leqslant |t| \leqslant 1\}$. We use an (inhomogeneous) dyadic partition of unity $1 = \psi(t) + \sum_{j\in\mathbb{N}} \phi(2^{-j}t)$ for all $t \in \mathbb{R}$, where $\phi \in \mathcal{C}_0^\infty(\mathcal{C})$ and $\psi \in \mathcal{C}_0^\infty(\mathcal{B})$. We write $\mathrm{Id} = \psi(L) + \sum_{j\in\mathbb{N}} \phi(2^{-jD}L)$, and we define the Littlewood–Paley operators by $\Delta_j := \phi(2^{-jD}L)$ and $S_j := \psi(2^{-jD}L)$ for $j \in \mathbb{N}$. We can similarly define homogeneous operators, $\dot{\Delta}_j$ and $\dot{S}_j$ using the decomposition $1 = \sum_{j\in\mathbb{Z}} \phi(2^{-j}t)$ for all $t \neq 0$. Then we consider the functions $(\Psi_j)_{j\in\mathbb{N}}$ and $(\Phi_j)_{j\in\mathbb{Z}}$ of $\mathcal{S}(G)$ such that $\Delta_j f = f * \Phi_j$ and $S_j f = f * \Psi_j$. By (1), we get $\Phi_j = 2^{jQ}\delta_{2^j}\Phi_0$ and $\Psi_j = 2^{jQ}\delta_{2^j}\Psi_0$. These truncation operators thus have very similar properties to those of the Euclidean case, so many classical results of $\mathbb{R}^d$ can be transposed exactly to this setting. In particular Littlewood–Paley operators allow to replace differentiation (via L) by multiplication by the size of the frequency (see [5,6]) as shown in the following Bernstein inequalities:

Lemma 2.2. *For any number $k \in \mathbb{R}$, there exists $C_k > 0$ so that, for any $(p,q) \in \mathbb{R}^2$ such that $q \geqslant p \geqslant 1$,*

$$C_k^{-1} 2^{jk} \|\Delta_j u\|_{L^p(G)} \leqslant \left\| L^{\frac{k}{D}} \Delta_j u \right\|_{L^p(G)} \leqslant C_k 2^{jk} \|\Delta_j u\|_{L^p(G)} \quad \text{and} \tag{3}$$

$$\left\| L^{\frac{k}{D}} \Delta_j u \right\|_{L^q(G)} \leqslant C_k 2^{jQ(\frac{1}{p}-\frac{1}{q})+jk} \|\Delta_j u\|_{L^p(G)}. \tag{4}$$

Moreover, for any smooth homogeneous function σ of degree m, there exists $C_{\sigma,m} > 0$ so that

$$\left\| \sigma\big(L^{\frac{1}{D}}\big) \Delta_j u \right\|_{L^p(G)} \leqslant C_{\sigma,m} 2^{jm} \|\Delta_j u\|_{L^p(G)}.$$

Proof. The proof is similar to the Euclidean case, so we shall prove (3) only. Let us write $L^{\frac{k}{D}} \Delta_j u = 2^{jk}(2^{-jD}L)^{\frac{k}{D}} \widetilde{\phi}(2^{-jD}L) \times \phi(2^{-jD}L)u$, with $\widetilde{\phi}$ a smooth function supported in a ring and equal to 1 near $\mathcal{C}$. We deduce in light of Theorem 2.1 that $L^{\frac{k}{D}} \Delta_j u = 2^{jk} \Delta_j u * \widetilde{\Phi}_k$, with $\widetilde{\Phi}_k = 2^{jQ} h_k(\delta_{2^j}\cdot)$ and $h_k \in \mathcal{S}(G)$. This ensures the right-hand side of (3) thanks to Young's inequalities. The proof of the left-hand side follows the same lines once we observe that $\Delta_j u = L^{-\frac{k}{D}} \Delta_j L^{\frac{k}{D}} u = 2^{-jk} \Delta_j L^{\frac{k}{D}} u * \widetilde{\Phi}_{-k}$. □

The dyadic decomposition allows to define a wide class of function spaces, like Besov spaces.

Definition 2.3. Let $s \in \mathbb{R}$ and (p,r) in $[1,\infty]^2$. We define the inhomogeneous and homogeneous Besov spaces by

$$B^s_{p,r}(G) = \left\{ u \in \mathcal{S}'(G),\ \|u\|_{B^s_{p,r}(G)} := \|S_0 u\|_{L^p} + \left\| \big(2^{js} \|\Delta_j u\|_{L^p(G)}\big) \right\|_{\ell^r(\mathbb{N})} < \infty \right\},$$

$$\dot{B}^s_{p,r}(G) = \left\{ u \in \mathcal{S}'(G),\ \dot{S}_j u \to 0 \text{ in } \mathcal{S}'(G) \text{ as } j \to -\infty \text{ and } \|u\|_{\dot{B}^s_{p,r}(G)} := \left\| \big(2^{js} \|\Delta_j u\|_{L^p(G)}\big) \right\|_{\ell^r(\mathbb{Z})} < \infty \right\}.$$

We point out that the ℓ^r norm is over $\mathbb{N}$ for inhomogeneous Besov spaces while it is over $\mathbb{Z}$ for homogeneous ones.

3. Refined inequalities

The scaling properties of the Littlewood–Paley decomposition allow to transpose classical proofs of refined inequalities of the Euclidean case: the method involves truncations into low and high frequencies and optimization of the cut-off parameter. We first give an estimate on the constant of continuity in the embedding of $\dot{H}^s(G)$ into $\dot{B}^{s-\frac{Q}{2}}_{\infty,\infty}(G)$ (see Proposition 1.41 in [2] for a proof).

Proposition 3.1. *There exists $C > 0$ such that for all $s < Q/2$,*

$$\|u\|_{\dot{B}^{s-\frac{Q}{2}}_{\infty,\infty}(G)} \leqslant C\left(\frac{Q}{2} - s\right)^{-\frac{1}{2}} \|u\|_{\dot{H}^s(G)} \quad \text{for all } u \in \dot{H}^s(G).$$

Refined Sobolev inequalities: The proofs of [11] (in $\mathbb{R}^d$) or [6] and [5] (in the Heisenberg group) give:

Corollary 3.1. *Let $1 \leqslant q < \infty$ and $0 < s < \frac{Q}{q}$; there exists $C > 0$ such that*

$$\|f\|_{L^p(G)} \leqslant C(p-q)^{\frac{1}{p}} \|f\|^{1-\frac{qs}{Q}}_{\dot{B}^s_{q,q}(G)} \|f\|^{\frac{qs}{Q}}_{\dot{B}^{s-\frac{Q}{q}}_{\infty,\infty}(G)}, \quad \frac{1}{p} = \frac{1}{q} - \frac{s}{Q}.$$

It is well known that the Sobolev space $H^{\frac{Q}{2}}(G)$ fails to be embedded in $L^\infty(G)$. However, the following Moser–Trudinger type inequalities derive easily from Corollary 3.1 (see Theorem 1.67 in [2]):

Corollary 3.2. *There exist $c, C > 0$ depending only on Q such that*

$$\forall f \in H^{\frac{Q}{2}}(G), \quad \int_G \left(\exp\left\{c\left(\frac{|f(x)|}{\|f\|_{H^{\frac{Q}{2}}}}\right)^2\right\} - 1\right) \mathrm{d}\mu(x) \leqslant C.$$

Gagliardo–Nirenberg inequalities: The maximal function on G is defined for all $f \in L^1_{loc}(G)$ by

$$\forall x \in G, \quad Mf(x) \stackrel{def}{=} \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \left|f(y)\right| \mathrm{d}\mu(y).$$

Then, for all $1 < p \leqslant \infty$, there exists a constant C such that $\|Mf\|_{L^p} \leqslant C\|f\|_{L^p}$. Moreover (see [2, Theorem 2.44]), for any $K \in L^1$ such that $\forall x \in G$, $|K(x)| \leqslant k(\rho(x))$ for some nonincreasing function $k : \mathbb{R}_+ \mapsto \mathbb{R}_+$, we have $|f \star K(x)| \leqslant \|K\|_{L^1(G)} Mf(x)$. Using those properties, along with (2), we can infer the following corollary (see for instance [2] for the Euclidean case or [5] for the Heisenberg group):

Corollary 3.3. *Let $1 \leqslant q, r \leqslant \infty$ and $0 < \sigma < s$; there exists $C > 0$ such that*

$$\forall f \in L^q(G) \cap \dot{W}^s_r(G), \quad \|f\|_{\dot{W}^\sigma_p(G)} \leqslant C\|f\|^\theta_{L^q(G)} \|f\|^{1-\theta}_{\dot{W}^s_r(G)} \quad \textit{with } \frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r},\ \theta = 1 - \frac{\sigma}{s}.$$

Refined Hardy inequalities: The result is the following:

Corollary 3.4. *Let $0 < s < \frac{Q}{q}$; there exists $C > 0$ such that*

$$\left(\int_G \frac{|f(x)|^q}{\rho(x)^{sq}} \mathrm{d}\mu(x)\right)^{\frac{1}{q}} \leqslant C\|f\|^{1-\frac{qs}{Q}}_{\dot{B}^s_{q,q}(G)} \|f\|^{\frac{qs}{Q}}_{\dot{B}^{s-\frac{Q}{q}}_{\infty,q}(G)}.$$

The proof consists in seeing the inequality as a weak Hölder inequality between ρ^{-s} and f. Thus since obviously $\rho^{-s} \in L^{Q/s,\infty}$, the result follows from weak Hölder inequalities along with (see [4])

$$\|f\|_{L^{p,q}(G)} \leqslant C\|f\|^{1-\frac{qs}{Q}}_{\dot{B}^s_{q,q}(G)} \|f\|^{\frac{qs}{Q}}_{\dot{B}^{s-\frac{d}{q}}_{\infty,q}(G)}.$$

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