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Functional Analysis

On some notions of spectra for quaternionic operators and for n-tuples of operators

Quelques notions de spectre pour les opérateurs quaternioniques et pour les n-uples d'opérateurs

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ABSTRACT

The S-spectrum has been introduced for the definition of the S-functional calculus that includes both the quaternionic functional calculus and a calculus for n-tuples of nonnecessarily commuting operators. The notion of right spectrum for right linear quaternionic operators has been widely used in the literature, especially in the context of quaternionic quantum mechanics. Moreover, several results in linear algebra, like the spectral theorem for quaternionic matrices, involve the right spectrum. In this Note we prove that the two notions of S-spectrum and of right spectrum coincide.

RÉSUMÉ

La notion de S-spectre a été introduite pour donner une définition de calcul formel utilisable pour des opérateurs linéaires quaternioniques et pour des n-uples d'opérateurs non nécessairement commutatifs. La notion de spectre à droite pour des opérateurs linéaires a été largement utilisée dans la littérature, particulièrement dans le cadre de la mécanique quantique. Par la suite, on a établi que de nombreux résultats d'algèbre linéaire, comme le théorème spectral pour les matrices quaternioniques, sont reliés au spectre à droite. Dans cette Note on démontre que les deux notions de S-spectre et de spectre à droite coïncident.

1. Introduction

In this Note we will consider some notions of spectra for quaternionic operators and for n-tuples of nonnecessarily commuting operators. In the first part of this Note, we will work with linear operators acting on a two sided quaternionic Banach space. Since we are in a noncommutative setting, we consider right linearity (this is the most used case in the literature), even though all our considerations can be repeated, interchanging everywhere left with right, for the case of left linear operators. In the case of quaternionic operators the most used notion of spectrum is that one of right spectrum. However, as we shall explain below, the right spectrum is not associated to a linear resolvent operator and this is a major problem if one needs to define a functional calculus. For the same reason, we can just speak of (right) *eigenvalues* but

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not of point, continuous and residual spectrum. Despite these disadvantages, the right spectrum has many applications in quaternionic Quantum Mechanics, as it is well exposed in the book of Adler [1], it is also widely studied in linear algebra, see e.g. [5,6] and the references therein, and it allows also to prove the quaternionic version of the spectral theorem. On the other hand, the S-functional calculus, which is the analog of the Riesz–Dunford functional calculus for quaternionic operators and for n-tuples of nonnecessarily commuting operators, involves the concept of S-spectrum and of S-resolvent operator, see the book [3]. In this paper we prove that the S-spectrum and the right spectrum coincide. This is an important result since it allows to relate a resolvent operator to the right spectrum. Furthermore, it shows that the S-spectrum used in the quaternionic functional calculus is the same spectrum considered in physics and in linear algebra. As a byproduct, all the results concerning the right spectrum are valid for the S-spectrum. In the second part of the paper, we consider n-tuples of nonnecessarily commuting linear operators $T_1, \dots, T_n$, acting on a real Banach space V. The operators T_ℓ, $\ell = 1, \dots, n$, can also be unbounded. As illustrated in [3], it is possible to define a functional calculus for n-tuples of operators by associating to $(T_1, \dots, T_n)$ the (right) linear operator $T = e_1T_1 + \cdots + e_nT_n$ acting on $V \otimes \mathbb{R}_n$ where $\mathbb{R}_n$ is the real Clifford algebra over the n imaginary units $e_1, \dots, e_n$. The functional calculus is based on the notion of S-spectrum which coincides with the right spectrum of the operator $e_1T_1 + \cdots + e_nT_n$.

2. The case of right linear quaternionic operators

We will denote a quaternion as $s = s_0 + is_1 + js_2 + ks_3$ and by $\mathbb{H}$ the real algebra of quaternions. Any nonreal quaternion s can be written as $s = \mathrm{Re}(s) + I|\mathrm{Im}(s)|$ where $\mathrm{Re}(s) = s_0$, $\mathrm{Im}(s)$ denote the real and the imaginary part of s, respectively, and I is an element of the 2-sphere $\mathbb{S}$ of purely imaginary quaternions of norm 1. By $T = T_0 + iT_1 + jT_2 + kT_3$ we denote a quaternionic operator, where T_ℓ, for $\ell = 0, 1, 2, 3$, are linear real operators. Let V be a two sided quaternionic Banach space. We denote by $\mathcal{B}(V)$ the left vector space of all right linear bounded operators on V. Note that in order to define a right linear operator on V it is sufficient to have just a right vector space structure on V, however, in order to have a vector space structure on $\mathcal{B}(V)$ it is necessary to assume that V is a two sided vector space. We denote by $\mathcal{BC}(V)$ the subset of $\mathcal{B}(V)$ consisting of quaternionic operators with commuting components.

The S-spectrum σ_S. For the definition of the quaternionic functional calculus a crucial fact is that the S-resolvent operator is a right linear operator and its existence is related to the S-spectrum:

Definition 2.1 *(The S-spectrum and the S-resolvent set).* Let $T \in \mathcal{B}(V)$. We define the S-spectrum $\sigma_S(T)$ of T as:

$$\sigma_S(T) = \{s \in \mathbb{H}:\ T^2 - 2s_0T + |s|^2\mathcal{I} \text{ is not invertible}\},$$

where $|s|^2 = s_0^2 + s_1^2 + s_2^2 + s_3^2$. The S-resolvent set $\rho_S(T)$ is defined by $\rho_S(T) = \mathbb{H} \setminus \sigma_S(T)$.

Let us recall that, given two quaternions s, q the relation $s \sim q$ if and only if $\mathrm{Re}(s) = \mathrm{Re}(q)$ and $|s| = |q|$ is an equivalence relation. The equivalence class $[s]$ consists of elements belonging to the 2-sphere associated to s, i.e. it consists of the elements of the form $\mathrm{Re}(s) + I|\mathrm{Im}(s)|$ where $I \in \mathbb{S}$.

Theorem 2.2 *(Structure of the S-spectrum). Let $T \in \mathcal{B}(V)$ and let $s = \mathrm{Re}(s) + I|\mathrm{Im}(s)| \in \sigma_S(T)$. Then all the elements of the 2-sphere $[s] = [\mathrm{Re}(s) + I|\mathrm{Im}(s)|]$ belong to $\sigma_S(T)$.*

Theorem 2.3 *(Compactness of S-spectrum). Let $T \in \mathcal{B}(V)$. Then the S-spectrum $\sigma_S(T)$ is a compact nonempty set.*

Denote by $(\bar{s}\mathcal{I})(v) := \bar{s}v$ the left multiplication by $\bar{s}$. The S-resolvent operator associated to $T \in \mathcal{B}(V)$, for $s \in \rho_S(T)$, is defined as $S^{-1}(s, T) := -(T^2 - 2s_0T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I})$.

The right spectrum σ_R. In the literature there are several works involving the notion of spectrum of a quaternionic right linear operator acting on a finite dimensional two sided vector space V. As it will be clear from the definition below, the notion of left spectrum is based on a resolvent operator while the notion of right spectrum is based on the notion of right eigenvalue.

Definition 2.4. Let $T : V \to V$ be a right linear quaternionic operator on a quaternionic Banach space V. We denote by $\sigma_L(T)$ the left spectrum of T related to the resolvent operator $(s\mathcal{I} - T)^{-1}$ that is $\sigma_L(T) = \{s \in \mathbb{H}:\ s\mathcal{I} - T \text{ is not invertible}\}$, where $(s\mathcal{I})(v) := sv$. We denote by $\sigma_R(T)$ the right spectrum of T that is $\sigma_R(T) = \{s \in \mathbb{H}:\ Tv = vs \text{ for } v \in V,\ v \neq 0\}$.

As it has been widely discussed in the literature, the notion of left spectrum is not very useful. As it has been shown in [3], the S-spectrum and the left spectrum are not, in general, related. The right spectrum is more useful and more studied. The right spectrum has a structure similar to the one of the S-spectrum, indeed whenever it contains an element s, it contains also the 2-sphere $[s]$. However the operator $\mathcal{I}s - T$, where $(\mathcal{I}s)(v) := vs$, is not a right linear operator; thus the notion of right spectrum is not associated to a linear resolvent operator and this represents a disadvantage since it prevents to define a functional calculus. We now prove our main result:

Theorem 2.5. *Let T be a right linear quaternionic operator. Then*

$$\sigma_S(T)=\sigma_R(T).$$

Proof. Proposition 3.1.6 in [3] shows that $\sigma_R(T)\cap\mathbb{R}=\sigma_S(T)\cap\mathbb{R}$.

Let us now consider the general case. Let $a+Ib\in\sigma_S(T)$, for $I\in\mathbb{S}$, $\mathbb{S}$ being the unit 2-sphere of purely imaginary quaternions. So $T^2-2aT+(a^2+b^2)\mathcal{I}$ is not invertible and there exists $v\neq 0$ such that $(T^2-2aT+(a^2+b^2)\mathcal{I})v=0$. If $Tv=v(a+Ib)$ then $a+Ib\in\sigma_R(T)$ and we are done; otherwise $Tv-v(a+Ib)$ is nonzero. We can write $(T^2-2aT+(a^2+b^2)\mathcal{I})v=0$ as $T(Tv-v(a+Ib))=(Tv-v(a+Ib))(a-Ib)$ which shows that $a-Ib$, and so the whole sphere $[a+Ib]$, belongs to $\sigma_R(T)$ in fact, whenever the right spectrum contains an element, it contains also the whole 2-sphere associated to it. Let us show the converse. Assume that $a+Ib\in\sigma_R(T)$, and so $Tv=v(a+Ib)$, for some $v\neq 0$. Then the element $a+Ib$ and the whole 2-sphere $[a+Ib]$ belong to $\sigma_S(T)$. In fact $T^2v-2aTv+(a^2+b^2)v=v(a+Ib)^2-2av(a+Ib)+(a^2+b^2)v=0$ and so $(T^2-2aT+(a^2+b^2)I)$ is not invertible. Thus $a+Ib\in\sigma_S(T)$ and this concludes the proof. □

In the paper [2] it is considered the F-spectrum, which is well defined just in the case the components of the quaternionic operator commute among themselves. The advantage of the F-spectrum is that it is easier to compute with respect to the S-spectrum indeed it can be computed by carrying out the computations in a complex plane, see [2]. Another interesting feature of the F-spectrum is that it allows the definition of a monogenic functional calculus.

Definition 2.6 *(The F-spectrum and the F-resolvent sets).* Let $T=T_0+iT_1+jT_2+kT_3\in\mathcal{BC}(V)$ and set $\overline{T}=T_0-iT_1-jT_2-kT_3$. We define the F-spectrum of T as:

$$\sigma_{\mathcal{F}}(T)=\{s\in\mathbb{H}:\ s^2\mathcal{I}-s(T+\overline{T})+T\overline{T}\ \text{is not invertible}\}.$$

The F-resolvent set of T is defined by $\rho_{\mathcal{F}}(T)=\mathbb{H}\setminus\sigma_{\mathcal{F}}(T)$.

Proposition 2.7. *Let T be a right linear quaternionic operator with commuting components. Then $\sigma_F(T)=\sigma_R(T)$.*

Proof. In the case T has commuting components thanks to Proposition 4.6 in [2] we have $\sigma_F(T)=\sigma_S(T)$. So by Theorem 2.5 we get the statement. □

As an immediate consequence of the previous results we have:

Corollary 2.8. *Let $\mathcal{BC}(V)$. Then $\sigma_F(T)=\sigma_R(T)=\sigma_S(T)$.*

Remark 1. As it is well known, the computation of the right spectrum is, in general, complicated, see [4–6] for the case of matrices. Our result shows that, whenever the components of a linear operator commute, then the computations can be done on a complex plane of the form $\mathbb{C}_I=\mathbb{R}+I\mathbb{R}$ thus making them easier, see [2].

Remark 2. Proposition 4.6 in [2] holds for n-tuples of commuting operators but the proof is valid also for quaternionic operators with commuting components.

3. The case of n-tuples of operators

We consider the real Clifford algebra $\mathbb{R}_n$ over n imaginary units $e_1,\dots,e_n$ satisfying the relations $e_ie_j+e_je_i=-2\delta_{ij}$. An element in the Clifford algebra will be denoted by $\sum_A e_Ax_A$ where $A=i_1\dots i_r$, $i_\ell\in\{1,2,\dots,n\}$, $i_1<\dots<i_r$ is a multi-index, $e_A=e_{i_1}e_{i_2}\dots e_{i_r}$ and $e_\emptyset=1$. By V we denote a Banach space over $\mathbb{R}$ and we set $V_n:=V\otimes\mathbb{R}_n$. We recall that V_n is a two sided Banach module over $\mathbb{R}_n$ and its elements are of the type $\sum_A v_A\otimes e_A$. A paravector is an element of the form $s=s_0+s_1e_1+\dots+s_ne_n$ and its squared norm is $|s|^2=s_0^2+s_1^2+\dots+s_n^2$. A paravector s can always be identified with an element $(s_0,s_1,\dots,s_n)$ in the Euclidean space $\mathbb{R}^{n+1}$. Right linear operators of the form $T=T_0+\sum_{j=1}^n e_jT_j$, acting on V_n, will be called operators in paravector form or paravector operators, for short. We denote by $\mathcal{B}^{0,1}(V_n)$ the set of bounded paravector operators and by $\mathcal{BC}^{0,1}(V_n)$ its subset of operators with commuting components. The conjugate of T is the operator $\overline{T}=T_0-\sum_{j=1}^n e_jT_j$. The notions of spectra in the case of paravector operators are as follows:

Definition 3.1. Let $T\in\mathcal{B}^{0,1}(V_n)$ and $s\in\mathbb{R}^{n+1}$. We define the S-spectrum $\sigma_S(T)$ of T as:

$$\sigma_S(T)=\{s\in\mathbb{R}^{n+1}:\ T^2-2s_0T+|s|^2\mathcal{I}\ \text{is not invertible}\}.$$

We denote by $\sigma_R(T)$ the right spectrum of T that is $\sigma_R(T)=\{s\in\mathbb{R}^{n+1}:\ Tv=vs\ \text{for}\ v\in V_n,\ v\neq 0\}$. Let $T\in\mathcal{BC}^{0,1}(V_n)$. We define the F-spectrum of T as:

$$\sigma_F(T)=\{s\in\mathbb{R}^{n+1}:\ s^2\mathcal{I}-s(T+\overline{T})+T\overline{T}\ \text{is not invertible}\}.$$

Proposition 3.2. *If $s \in \mathbb{R}^{n+1}$ belongs to $\sigma_R(T)$ then the $(n-1)$-sphere $[s]$ belongs to $\sigma_R(T)$.*

Proof. Since $\sigma_R(T) \subset \mathbb{R}^{n+1}$ if $Tv = vs$ then for any $p \neq 0$ we have $T(vp) = vsp = vp(p^{-1}sp)$. □

We point out the F-spectrum is well defined only for $T \in \mathcal{BC}^{0,1}(V_n)$. With obvious modifications of the proofs of Theorem 2.5 and of Proposition 2.7 we have the following results:

Theorem 3.3. *Let $T \in \mathcal{B}^{0,1}(V_n)$. Then $\sigma_S(T) = \sigma_R(T)$.*

Proposition 3.4. *Let $T \in \mathcal{BC}^{0,1}(V_n)$. Then $\sigma_F(T) = \sigma_R(T)$.*

So we observe that the three notions of spectra coincide in the case the paravector operator T has commuting components $\sigma_F(T) = \sigma_S(T) = \sigma_R(T)$. It is also possible to define the left spectrum of T but it is not related to the previous spectra.

References

[1] S. Adler, Quaternionic Quantum Field Theory, Oxford University Press, 1995.
[2] F. Colombo, I. Sabadini, The $\mathcal{F}$-spectrum and the $\mathcal{SC}$-functional calculus, Proc. Roy. Soc. Edinburgh Sect. A 142 (2012) 1–22.
[3] F. Colombo, I. Sabadini, D.C. Struppa, Noncommutative Functional Calculus. Theory and Applications of Slice Hyperholomorphic Functions, Progress in Mathematics, vol. 289, Birkhäuser/Springer Basel AG, Basel, 2011, vi+221 pp.
[4] S. De Leo, G. Scolarici, Right eigenvalue equation in quaternionic quantum mechanics, J. Phys. A 33 (2000) 2971–2995.
[5] D.R. Farenick, B.A.F. Pidkowich, The spectral theorem in quaternions, Linear Algebra Appl. 371 (2003) 75–102.
[6] F. Zhang, Quaternions and matrices of quaternions, Linear Algebra Appl. 251 (1997) 21–57.