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Number Theory/Algebraic Geometry

## On uniform boundedness of a rational distance set in the plane

*Sur une borne uniforme d'un ensemble de distances rationnelles sur un plan*Mehdi Makhul, Jafar Shaffaf<sup>1</sup>

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## ABSTRACT

A rational set in the plane is a point set with all its pairwise distances rational. Ulam asked in 1945 if there is an everywhere dense rational set. Solymosi and de Zeeuw proved that every rational distance subset of the plane has only finitely many points in common with an irreducible algebraic curve defined over  $\mathbb{R}$  unless the curve is a line or circle. As an application of uniformity conjecture in arithmetic algebraic geometry which is a consequence of Lang conjecture we prove that if  $S$  is an infinite rational distance subset of the plane that has only finitely many points on any line then there is a uniform bound (independent of  $S$ ) on the number of these points.

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## R É S U M É

Un ensemble sur un plan est appelé rationnel si la distance entre tous ses points est rationnelle. Une question posée par Ulam en 1945 demande s'il existe un ensemble rationnel et partout dense sur un plan. Solymosi et de Zeeuw ont démontré que l'intersection de toute courbe algébrique irréductible définie sur  $\mathbb{R}$  avec tout ensemble rationnel sur un plan est un ensemble fini sauf si la courbe est une ligne droite ou un cercle. Comme application de la conjecture d'uniformité en géométrie arithmétique, une conséquence de la conjecture de Lang, nous démontrons que si  $S$  est un ensemble rationnel et infini sur un plan dont l'intersection avec toute ligne droite est un ensemble fini alors il existe une borne uniforme sur le cardinal de ces ensembles d'intersection.

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## 1. Introduction

We define a rational set to be a set  $S$  in the plane (by plane we mean  $\mathbb{R}^2$ ) such that the distance between any two elements is a rational number. On any line, one can easily find an infinite rational set that is in fact dense. It is an easy exercise to find an everywhere dense rational subset of the unit circle. However it is not known if there is a rational set with 8 points in general position, i.e. no 3 on a line, no 4 on a circle. In 1945, Anning and Erdős [1] proved that any infinite integral set, i.e. where all distances are integers, must be contained in a line. In 1945, when Ulam heard Erdős's simple proof [1] of his theorem with Anning, he said that he believed there is no everywhere dense rational set in the plane, see Problem III.5. in [14]. Erdős conjectured that an infinite rational set must be very restricted, but that it is probably a very deep problem [8].

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Using Mordell conjecture proved by Faltings, J. Solymosi and F. de Zeeuw proved [13] that lines and circles are the only irreducible algebraic curves that contain an infinite rational set. They also showed that if a rational set  $S$  has infinitely many points on a line or on a circle, then all but 4 respectively 3 points of  $S$  are on the line or on the circle. This answers questions of Guy, Problem D20 in [9], and Pach, Section 5.11 in [2].

There were attempts to find rational sets on parabolas [3,7], and there were some results on integral sets, in particular bounds were found on the diameter of integral sets [12]. Recently Kreisel and Kurz [11] found an integral set with 7 points in general position.

For a rational distance subset of the plane intersecting any line in finitely many points, we prove that there is a uniform bound on the number of these intersections. Our main tool is a conditional uniform boundedness theorem on the number of rational points of algebraic curves of genus  $g \geq 2$  on a fixed number field  $K$  [6] proved by L. Caporaso, J. Harris and B. Mazur. Their result depends on the weak Lang conjecture.

For a short survey we refer to the article by L. Caporaso [4] and the article [5].

In fact in this Note we use the rational set to give a lower bound for uniform bounded theorem.

**Weak Lang Conjecture.** *Let  $X$  be a variety of general type defined over a number field  $K$ . It was conjectured by S. Lang that the set of rational points  $X(K)$  is not Zariski dense in  $X$ .*

In the paper [6] of L. Caporaso, J. Harris and B. Mazur the following result is proved:

**Uniform Bound Theorem 1.** *The weak Lang conjecture implies that, for every number field  $K$  and for every  $g \geq 2$ , there exists a number  $B(K, g)$  such that no curve of genus  $g$  defined over  $K$  has more than  $B(K, g)$  points defined over  $K$ .*

One can view [6] either as a way to construct a counter example to weak Lang conjecture or as a powerful implication of weak Lang conjecture. Our result can be viewed in either light as well. For example, if the statement of the (conditional) Theorem 3.1 were proved to fail, then both uniformity conjecture and weak Lang conjecture would be proved to fail.

## 2. Preliminaries and known results

Rationality of distances in  $\mathbb{R}^2$  is clearly preserved by translations, rotations, and uniform scaling  $((x, y) \rightarrow (\lambda x, \lambda y)$  with  $\lambda \in \mathbb{Q}$ ). We call a transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  a similarity transformation if it can be written as the composition of translations, rotations and uniform scaling. More surprisingly, rational sets are preserved under certain central inversions (an inversion with respect to a point in the rational set and with a rational radius), more precisely we bring the following lemma from [13]:

**Lemma 2.1.** *If we apply a central inversion to a rational set  $S$ , with center a point  $x \in S$  and rational radius, then the image of  $S - \{x\}$  is a rational set.*

We use Lemma 2.1 in Corollary 3.3 to deduce the uniformity bound for circles rather than lines under the assumption that there is only finitely many points of the rational set on each circle.

A priori, points in a rational set could take any form. However, after moving two of the points to two fixed rational points by translating, rotating and scaling, the points are almost rational points. The following simple lemma is illuminating in itself to better understand the structure of a rational distance set. As far as we know, it was proved first by Kemnitz [10].

**Lemma 2.2.** *For any rational set  $S$  there is a square free integer  $k$  such that if a similarity transformation  $T$  transforms two points of  $S$  into  $(0, 0)$  and  $(1, 0)$  then any point in  $T(S)$  is of the form  $(r_1, r_2\sqrt{k})$ ,  $r_1, r_2 \in \mathbb{Q}$ .*

**Proof.** Let  $(0, 0)$  and  $(1, 0)$  and  $(x, y)$  be members of the rational set  $S$  then we have  $x^2 + y^2 = q_1^2$  and  $(x - 1)^2 + y^2 = q_2^2$  such that  $q_1, q_2 \in \mathbb{Q}$ . By solving these two equations we have  $x = \frac{q_1^2 - q_2^2 + 1}{2}$  so  $x$  is a rational number. By substitution of the value of  $x$  and a simple manipulation for  $y$  it can be easily shown that  $y = r\sqrt{k}$  where  $r \in \mathbb{Q}$  and  $k$  is square free. For uniqueness of  $k$  let  $p_1 = (r_1, r_2\sqrt{k})$  and  $p_2 = (r'_1, r'_2\sqrt{k'})$  be in the rational set  $S$  hence the distance between  $p_1$  and  $p_2$  is a rational number. By writing the square of the distance between two points  $p_1$  and  $p_2$  it follows that the number  $2r_2r'_2\sqrt{kk'}$  should be a rational number and hence  $\sqrt{kk'}$  is a rational number and since  $k$  and  $k'$  are square free therefore we have  $k = k'$ .  $\square$

**Remark 1.** One can show that any curve of degree  $d$  containing at least  $\frac{d(d+3)}{2}$  points from  $T(S)$  in general position is defined over  $\mathbb{Q}(\sqrt{k})$ . This can be done by explicit calculation of the coefficients of the equation defining  $C$  and using the above lemma to find that those coefficients are from  $\mathbb{Q}(\sqrt{k})$ .

Note that we have the following two important theorems from [13]:

**Theorem 2.3.** (See [13].) *Every rational subset of the plane has only finitely many points in common with an algebraic curve defined over  $\mathbb{R}$ , unless the curve has a component which is a line or circle.*

**Theorem 2.4.** (See [13].) *If a rational set  $S$  has infinitely many points on a line (resp. circle) then all but 4 (resp. 3) points of  $S$  are on the line (circle).*

### 3. The main result

Now that the previous two theorems give a complete description of rational distance sets with infinitely many points on a line, we proceed with the remaining case.

Our main result is the following:

**Theorem 3.1.** *Assume the Weak Lang Conjecture and let  $S$  be an infinite rational subset of the plane which has finite intersection with each line. Then there exists an integer number  $B(S)$  such that for each line  $L$  in the plane the intersection  $S \cap L$  has at most  $B(S)$  points. Moreover  $B(S) \leq B(\mathbb{Q}, 2)$ , in particular there is an upper bound for  $B(S)$  independent of  $S$ .*

For the proof we use uniform bound theorem which is heavily relied on Weak Lang Conjecture.

If the Kodaira dimension of a variety is equal to its dimension the variety is called to be of general type. For the case of our interest, the curves, being of general type is equivalent that the curve has genus greater than or equal to 2.

**Weak Lang Conjecture.** *If  $X$  is a variety of general type defined over a number field  $K$  then the set  $X(K)$  of  $K$ -rational points of  $X$  is not Zariski dense.*

An important arithmetic consequence of weak Lang conjecture is the uniform bound theorem proved by Caporaso, Harris and Mazur [6].

**Theorem 3.2 (Uniform bound).** *Assume the weak Lang conjecture and let  $K$  be a number field and  $g \geq 2$  an integer. There exists an integer  $B(K, g)$  such that no smooth curve of genus  $g$  defined over  $K$  has more than  $B(K, g)$  rational points.*

**Proof of Theorem 3.1.** Let  $S$  be an infinite rational set in the plane and assume that  $(0, 0) \in S$ , the proof proceeds by contradiction. Suppose the statement of the theorem is not true. So for any integer number  $n$  there exists a line  $L$  in the plane such that we have  $|S \cap L| \geq n$ . Now let  $n$  be sufficiently large so that  $n \gg B(\mathbb{Q}, 2)$ . By a rotation we can assume that the line  $L$  is the  $x$ -axis and we have that  $S \cap L = \{(x_1, 0), \dots, (x_n, 0)\}$  and note that since  $(0, 0) \in S$  each  $x_j$  is rational. By choosing the points  $(\alpha_i, \beta_i) \in S$  for  $i = 1, 2, 3$  we can construct a smooth hyperelliptic curve associated to the equation

$$y^2 = \prod_{i=1}^3 ((x - \alpha_i)^2 + \beta_i^2).$$

Note that for smoothness it is enough to choose the distinct points  $(\alpha_i, \beta_i) \in S$  with nonzero  $\beta_i$  so that the roots of the right-hand side are distinct. It is easy to see that the genus of this hyperelliptic curve is 2. Now by substituting each rational number  $x_j$  in the right-hand side of the equation of hyperelliptic curve, the expressions  $((x - \alpha_i)^2 + \beta_i^2)$  are the squares of the distances between points in  $S$ , hence they are squares of rational numbers, for all  $j = 1, \dots, n$  and all  $i$  and therefore the coordinate  $y$  is a rational number. Hence we have at least  $n$  rational points  $\{(x_j, y_j)\}_{j=1}^n$  on the hyperelliptic curve and according to the choice of  $n$  this is a contradiction by uniform bound theorem for  $g = 2$ .  $\square$

**Corollary 3.3.** *Let  $S$  be an infinite rational set in the plane which has finite intersection with every circle in the plane. Then there exist an integer number  $B(S)$  such that for each circle  $C$  in the plane the intersection  $S \cap C$  has at most  $B(S)$  points.*

**Proof.** By application of a central inversion with respect to a point in  $S$ , say  $(0, 0)$  (which can be chosen to be in  $S$ ) the circles transform to lines and according to Lemma 2.1 the image of a rational set under a central inversion with rational radius is a rational set. Now the hypothesis of the previous theorem is satisfied and therefore the corollary is proved by the Theorem 3.1.  $\square$

One can pose the following questions:

**Question 1.** Let  $S$  be an infinite rational subset of the plane which has finite intersection with each plane curve of fixed degree  $d$ . Does there exist an integer number  $B_d(S)$  such that for each curve  $C$  of degree  $d$  in the plane the intersection  $S \cap C$  has at most  $B_d(S)$  points?

**Question 2.** Let  $C$  be an irreducible algebraic curve which is not a line or a circle and let  $S$  be an infinite rational set in the plane. Does there exist an integer number  $B(S)$  such that for each similarity transformation  $T$  of the plane the set  $S \cap T(C)$  has at most  $B(S)$  points?

In a future work we answer the above questions affirmatively.

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