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Partial Differential Equations

An analytical solution of the shallow water system coupled to the Exner equation

Une solution analytique du système de Saint-Venant couplé à l'équation d'Exner

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A B S T R A C T

In this Note, an exact smooth solution for the equations modeling the bedload transport of sediment in shallow water is presented. This solution is valid for a large family of sedimentation laws which are widely used in erosion modeling such as the Grass model or those of Meyer-Peter and Müller. One of the main interests of this solution is the derivation of numerical benchmarks to valid the approximation methods.

R É S U M É

Cette Note présente une solution analytique pour le système modélisant le transport de sédiments par le charriage. Cette solution est valable pour une grande famille de lois sédimentaires comme le modèle de Grass ainsi que celui de Meyer-Peter et Müller. Ce résultat est utile pour la validation des schémas numériques.

1. Introduction

Soil erosion is a consequence of the movements of sediments due to mechanical actions of flows. In the context of bedload transport, a mass conservation law, also called the Exner equation [5], is used to update the bed elevation. This equation is coupled with the shallow water equations describing the overland flows (see [2] and references therein) as follows:

$$\partial_t h + \partial_x(hu) = 0, \tag{1}$$

$$\partial_t(hu) + \partial_x\left(hu^2 + gh^2/2\right) + gh\partial_x z_b = 0, \tag{2}$$

$$\partial_t z_b + \partial_x q_b = 0, \tag{3}$$

where h is the water depth, u the flow velocity, z_b the thickness of sediment layer which can be modified by the fluid and g the acceleration due to gravity. The variable hu is also called water discharge and noted by q. Finally, q_b is the volumetric

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bedload sediment transport rate. Its expressions are usually proposed for granular non-cohesive sediments and quantified empirically [6,8,3].

Many numerical schemes have been developed to solve system (1)–(3) (see [3] and references therein). The validation of such schemes by an analytical solution is a simple way to ensure their working. Nevertheless, analytical solutions are not proposed in the literature. Up to our knowledge, asymptotic solutions, derived by Hudson in [7], are in general adopted to perform some comparisons with approximated solutions. The solutions are derived for Grass model [6], *i.e.* $q_b = A_g u^3$, when the interaction constant A_g is smaller than 10^{-2}. In this paper, we propose a non-obvious analytical solution in the steady state condition of flow.

2. Solution of the equations

In order to obtain an analytic solution, we consider q_b as a function of the dimensionless bottom shear stress τ_b^* (see [3]). By using the friction law of Darcy and Weisbach, τ_b^* is given by

$$\tau_b^* = \frac{f u^2}{8(s-1)g d_s},$$

where f is the friction coefficient, $s = \rho_s/\rho$ the relative density of sediment in water and d_s the diameter of sediment. The formula of q_b is usually expressed under the form

$$q_b = \kappa\left(\tau_b^* - \tau_{cr}^*\right)^p \sqrt{(s-1)g d_s^3}, \tag{4}$$

where τ_{cr}^* is the threshold for erosion, κ an empirical coefficient and p an exponent which is often fixed to $3/2$ in many applications. The expression (4) can be written in the simple form

$$q_b = A u_e^{2p}, \tag{5}$$

where the effective velocity u_e and the interaction coefficient A are defined by

$$\begin{cases} u_e^2 = u^2 - u_{cr}^2, \\ u_{cr}^2 = \tau_{cr}^* \left[\dfrac{f}{8(s-1)g d_s}\right]^{-1}, \\ A = \kappa \left[\dfrac{f}{8(s-1)g d_S}\right]^p \sqrt{(s-1)g d_s^3}. \end{cases} \tag{6}$$

Remark. The Grass model [6] is one of the simplest cases by using $p = 3/2$, $\tau_{cr}^* = 0$ and an empirical coefficient A_g instead of A. The Meyer-Peter and Müler model [8] is one of the most applied by using $p = 3/2$, $\kappa = 8$, $\tau_{cr}^* = 0.047$. The following result is valid for all models rewriting in form (5)–(6).

Proposition 2.1. *Assume that q_b is defined by* (4)*. For a given uniform discharge q such that $\tau_b^* > \tau_{cr}^*$, system* (1)–(3) *has the following analytical unsteady solution*

$$\begin{cases} u_e^2 = \left[\dfrac{\alpha x + \beta}{A}\right]^{1/p}, \\ u = \sqrt{u_e^2 + u_{cr}^2}, \quad h = q/u, \\ z_b^0 = -\dfrac{u^3 + 2gq}{2gu} + C, \\ z_b = -\alpha t + z_b^0, \end{cases} \tag{7}$$

where α, β, C are constants and A, u_{cr} are defined by (6)*.*

Proof. We are here concerned for the smooth solution. In view of the assumption $hu = q = \text{cst}$, Eqs. (1)–(3) reduce to

$$\partial_t h = 0,$$

$$\partial_x\left(q^2/h\right) + gh\partial_x H = 0, \tag{8}$$

$$\partial_t H + \partial_x q_b = 0, \tag{9}$$

where $H = h + z$ is the free surface elevation. Differentiating Eq. (8) with respect to t and then Eq. (9) with respect to x, we obtain

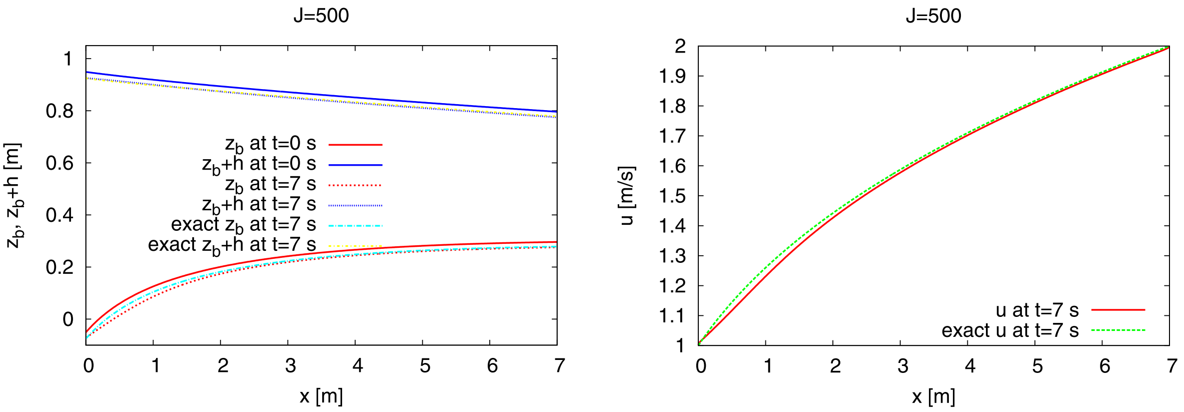

Fig. 1. Comparison between the exact solution and the relaxation method for: the water height and the topography (left) and the velocity (right).

$$\partial_{xt} H = 0,$$
$$\partial_x^2 q_b = 0. \tag{10}$$

Note that we can write $q_b = q_b(h, q)$ to have $\partial_t q_b = \partial_h q_b \partial_t h + \partial_q q_b \partial_t q = 0$, so q_b is not time-depending. Thanks to (10), the expression of q_b is obtained under the form

$$q_b = \alpha x + \beta, \tag{11}$$

where α and β are constant. From (3), we obtain $\partial_t z_b = -\partial_x q_b = -\alpha$ to write

$$z_b = -\alpha t + z_b^0(x). \tag{12}$$

Moreover, from (5) we deduce the effective velocity as follows:

$$u_e^2 = \left[\frac{\alpha x + \beta}{A}\right]^{1/p}.$$

Plugging (12) into the momentum equation (8) and using a direct calculation, we have

$$\partial_x z_b^0 = \left[\frac{q}{u^2} - \frac{u}{g}\right]\partial_x u \quad \Rightarrow \quad z_b^0 = -\frac{u^3 + 2gq}{2gu} + C$$

which concludes the proof. □

Remark. As h and u are stationary, the initial condition of (7) is (h, u, z_b^0). Moreover, the solution (h, u) applied to the Grass model is also an analytical solution of the shallow water equations with the variable topography z_b^0. Concerning the shallow water model, other solutions can be found in [4].

3. Numerical experiments

In this section, we consider the analytical solution (7) applied to the Grass model with $q = 1$, $A_g = \alpha = \beta = 0.005$ and $C = 1$. A relaxation solver is applied to approximate the solution of the model. We will not give here the details of the relaxation solver (for details see [1]), but just the relaxation model for Eqs. (1)–(3). Thus, we solve the following relaxation system:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + \pi\right) + gh\partial_x z_b = 0, \\ \partial_t \pi + u\partial_x \pi + \dfrac{a^2}{h}\partial_x u = 0, \\ \partial_t z_b + \partial_x q_r = 0, \\ \partial_t q_r + \left(\dfrac{b^2}{h^2} - u^2\right)\partial_x z_b + 2u\partial_x q_r = 0, \end{cases}$$

that is completed with $\pi = gh^2/2$ and $q_r = q_b$ at the equilibrium. Fig. 1 presents the numerical result with $J = 500$ space cells, a CFL fix condition of 1 and $T = 7$ s. We only notice little difference on the velocity, near the inflow boundary.

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