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Variation on a theme by Bobylëv and Villani

Variations sur un théme de Bobylëv et Villani

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ABSTRACT

It is shown that the collisional gain operator for a Maxwell gas does not increase the Fisher information. Our proof is a variant of the one given by Villani in 1998 [2], but it is shorter and based on Fourier techniques rather than direct estimates. The method we use also applies to general (non-symmetric) Wild convolutions.

RÉSUMÉ

On montre que le terme de gain de l'opérateur de collision pour un gaz de molécules Maxwelliennes n'induit pas d'augmentation de l'information de Fisher. Notre preuve est une variante de celle donnée par Villani dans 1998 [2], plus courte et basée sur des techniques de type Fourier plutôt que sur des estimations directes. La méthode utilisée s'applique aussi au cas des convolutions de Wild générales (non symétriques).

1. Introduction

The collisional gain operator for Maxwellian molecules in $\mathbb{R}^N$ is defined by

$$Q_+[f,g](v) = \iint_{\mathbb{R}^N\times\mathbb{S}^{N-1}} B(n\cdot\hat{q}) f(v') g(v'_*)\,\mathrm{d}v_*\,\mathrm{d}\sigma(n). \tag{1}$$

In the homogeneous Boltzmann equation $\partial_t f = Q_+[f,f] - f$, the difference $Q_+[f,f] - f$ accounts for the changes in the velocity distribution f due to binary particle collisions. In (1), the cross section $B : [-1,1] \to \mathbb{R}_{\geqslant 0}$ determines the frequency at which collisions between particles of velocities v' and v'_* occur, and these pre-collisional velocities are related to the post-collisional ones, v and v_*, by

$$v' = \frac{1}{2}\big(v + v_* + |q|n\big), \qquad v'_* = \frac{1}{2}\big(v + v_* - |q|n\big), \quad \text{with } q = v - v_* \text{ and } \hat{q} = q/|q|.$$

In Maxwellian gases, B depends on $n\cdot\hat{q}$ but not on $|v - v_*|$, and

$$\int_{\mathbb{S}^{N-1}} B(\mathbf{e}\cdot n)\,\mathrm{d}\sigma(n) = 1 \quad \text{for every unit vector } \mathbf{e} \in \mathbb{R}^N. \tag{2}$$

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Under the additional assumption that B is even, $B(s)=B(-s)$, it has been shown by Villani [2] that Q_+ does not increase the Fisher information, defined on probability densities f by

$$\mathcal{F}[f]=\int_{\mathbb{R}^N}\frac{|\nabla f(v)|^2}{f(v)}\,\mathrm{d}v=4\int_{\mathbb{R}^N}\left|\nabla\sqrt{f}(v)\right|^2\mathrm{d}v.$$

More precisely, it has been shown that $\mathcal{F}[Q_+[f,g]]\leqslant(\mathcal{F}[f]+\mathcal{F}[g])/2$. Below, we prove that the hypothesis $B(s)=B(-s)$ can be removed at the price of replacing the original estimate by

$$\mathcal{F}\big[Q_+[f,g]\big]\leqslant\frac{1+\lambda_B}{2}\mathcal{F}[f]+\frac{1-\lambda_B}{2}\mathcal{F}[g]\quad\text{with } \lambda_B:=\int_{\mathbb{S}^{N-1}}(\mathbf{e}\cdot n)B(\mathbf{e}\cdot n)\,\mathrm{d}\sigma(n)\in(-1,1).\tag{3}$$

For even B, we have $\lambda_B=0$ and thus recover the estimate [2]. Our main contribution in this note, however, is a novel, concise derivation of the following representation formula,

$$\nabla Q_+[f,g](v)=\frac{1}{2}\iint_{\mathbb{S}^{N-1}\times\mathbb{R}^N}B(\hat{q}\cdot n)\big\{Y'_++\mathrm{P}_{n,q}\big[Y'_-\big]\big\}\,\mathrm{d}\sigma(n)\,\mathrm{d}v_*,\quad Y'_\pm:=\nabla f\big(v'\big)g\big(v'_*\big)\pm f\big(v'\big)\nabla g\big(v'_*\big).\tag{4}$$

The "projection" operator P is defined for $a,b,x\in\mathbb{R}^N$ by

$$\mathrm{P}_{a,b}[x]=(\hat{a}\cdot\hat{b})x-(\hat{a}\wedge\hat{b})\cdot x,\quad\text{with } v\wedge w=vw^T-wv^T,\tag{5}$$

and $\hat{a}=a/|a|$ denotes the normalization of a vector $a\in\mathbb{R}^N$. Notice that $v\wedge w$ is an anti-symmetric $N\times N$-matrix, and in particular, in dimension $N=3$, one has $(\hat{a}\wedge\hat{b})\cdot x=(\hat{a}\times\hat{b})\times x$. Formula (4) — for symmetric B — is at the heart of Villani's original proof, where it is obtained from integration by parts and geometric considerations. Below, we prove (4) in one line by Fourier methods.

Notation. Inside integrals, f, f', g_*, g'_* abbreviate $f(v)$, $f(v')$, $g(v_*)$, $g(v'_*)$, and Q_+ abbreviates $Q_+[f,g]$.

2. Fourier representation

Our starting point is the following famous identity by Bobylëv [1]:

Lemma 2.1. *Given two probability densities f and g, then*

$$\widehat{Q}_+[f,g](\xi)=\int_{\mathbb{S}^{N-1}}B(n\cdot\hat{\xi})\hat{f}(\xi_+)\hat{g}(\xi_-)\,\mathrm{d}\sigma(n)\quad\textit{with } \xi_\pm=\frac{1}{2}\big(\xi\pm|\xi|n\big).\tag{6}$$

The key observation is that representation (6) in combination with the elementary relation (7) below — which admits a one-line proof — yields the Fourier analogue of (4).

Lemma 2.2. *For arbitrary $\xi\in\mathbb{R}^N$, and with $\xi_\pm$ defined in* (6)*,*

$$(\mathbf{1}+\mathrm{P}_{n,\xi})[\xi_+]+(\mathbf{1}-\mathrm{P}_{n,\xi})[\xi_-]=2\xi.\tag{7}$$

Proof. On one hand, $\xi_++\xi_-=\xi$ follows directly from (6). And on the other hand, also

$$\mathrm{P}_{n,\xi}[\xi_+-\xi_-]=\mathrm{P}_{n,\xi}\big[|\xi|n\big]=|\xi|\big((n\cdot\hat{\xi})n-n(n\cdot\hat{\xi})+\hat{\xi}(n\cdot n)\big)=|\xi|\hat{\xi}=\xi,$$

since $n\cdot n=1$, and $\hat{\xi}:=\xi/|\xi|$ by definition. □

Inserting the relation (7) under the integral in (6) gives

$$\widehat{\nabla Q}_+(\xi)=i\xi\,\widehat{Q}_+(\xi)=\frac{1}{2}\int_{\mathbb{S}^{N-1}}B(\hat{\xi}\cdot n)\big((\mathbf{1}+\mathrm{P}_{n,\xi})[i\xi_+]+(\mathbf{1}-\mathrm{P}_{n,\xi})[i\xi_-]\big)\hat{f}(\xi_+)\hat{g}(\xi_-)\,\mathrm{d}\sigma(n).\tag{8}$$

We shall now show that the Fourier transform of (8) is (4). Substituting

$$\hat{f}(\xi_+)=\int_{\mathbb{R}^N}e^{-iv\cdot\xi_+}f(v)\,\mathrm{d}v,\qquad i\xi_+\hat{f}(\xi_+)=\int_{\mathbb{R}^N}e^{-iv\cdot\xi_+}\nabla f(v)\,\mathrm{d}v,$$

and respective expressions for $\hat{g}(\xi_-)$, $i\xi_-\hat{g}(\xi_-)$ under the integral in (8), gives, with $Y_\pm := \nabla f g_* \pm f \nabla g_*$,

$$\widehat{\nabla Q}_+(\xi) = i\xi\,\widehat{Q}_+(\xi) = \frac{1}{2}\iiint\limits_{\mathbb{S}^{N-1}\times\mathbb{R}^N\times\mathbb{R}^N} B(\hat{\xi}\cdot n)\big\{Y_+ + \mathrm{P}_{n,\xi}[Y_-]\big\}e^{-i(v\cdot\xi_+ + v_*\cdot\xi_-)}\,\mathrm{d}\sigma(n)\,\mathrm{d}v\,\mathrm{d}v_*$$
$$= \frac{1}{2}\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\bigg(\int\limits_{\mathbb{S}^{N-1}} e^{i|\xi|(q\cdot n)}B(\hat{\xi}\cdot n)\big\{Y_+ + \mathrm{P}_{n,\xi}[Y_-]\big\}\,\mathrm{d}\sigma(n)\bigg)e^{-i\xi\cdot(v+v_*)/2}\,\mathrm{d}v\,\mathrm{d}v_*.$$

Next, we apply a particular change of variables — which has been designed by Bobylëv [1] — inside the n-integral to exchange the roles of ξ and q. For the corresponding treatment of the projection operator, we need

Lemma 2.3. *For arbitrary vectors $q, \xi \in \mathbb{R}^N \setminus \{0\}$, and for any measurable function $A : [-1, 1] \times [-1, 1] \to \mathbb{R}$, one has*

$$\int\limits_{\mathbb{S}^{N-1}} A(\hat{q}\cdot n, \hat{\xi}\cdot n)\hat{q}\wedge n\,\mathrm{d}\sigma(n) = -\int\limits_{\mathbb{S}^{N-1}} A(\hat{\xi}\cdot n, \hat{q}\cdot n)\hat{\xi}\wedge n\,\mathrm{d}\sigma(n). \tag{9}$$

In fact, both (matrix-valued) integrals are multiples of $\xi \wedge q$, and vanish if ξ and q are linearly dependent.

Before proving (9), we show that it indeed concludes the calculation started above. First, observe that (notice the change of order in the subscripts)

$$\mathrm{P}_{n,\xi} = (n\cdot\hat{\xi})\mathbf{1} - n\wedge\hat{\xi} \quad \text{and} \quad \mathrm{P}_{q,n} = (n\cdot\hat{q})\mathbf{1} + n\wedge\hat{q}.$$

We substitute (9) under the n-integral above and observe that its value does not change upon replacing n by its mirror image in the hyperplane orthogonal to $\hat{\xi}$–$\hat{q}$,

$$\widehat{\nabla Q}_+(\xi) = \frac{1}{2}\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\bigg(\int\limits_{\mathbb{S}^{N-1}} e^{i|q|(\xi\cdot n)}B(\hat{q}\cdot n)\big\{Y_+ + \mathrm{P}_{q,n}[Y_-]\big\}\,\mathrm{d}\sigma(n)\bigg)e^{-i\xi\cdot(v+v_*)/2}\,\mathrm{d}v\,\mathrm{d}v_*$$
$$= \frac{1}{2}\iint\limits_{\mathbb{S}^{N-1}\times\mathbb{R}^N\times\mathbb{R}^N} B(\hat{q}\cdot n)\big\{Y_+ + \mathrm{P}_{q,n}[Y_-]\big\}\,\mathrm{d}\sigma(n)e^{-i\xi\cdot v'}\,\mathrm{d}v\,\mathrm{d}v_*.$$

Formula (4) is now obtained by performing a change of variables $(v, v_*) \leftrightarrow (v', v'_*)$ under the integral. This substitution is of determinant one, it changes $Y_\pm$ into $Y'_\pm$, and it exchanges $\hat{q}$ with n as desired.

Proof of Lemma 2.3. Let $X \subset \mathbb{R}^N$ be the subspace spanned by ξ and q. Denote its orthogonal complement by $X^\perp$. We start by proving the second claim, namely that

$$I := \int\limits_{\mathbb{S}^{N-1}} A(\hat{q}\cdot n, \hat{\xi}\cdot n)\hat{q}\wedge n\,\mathrm{d}\sigma(n) \quad \text{and} \quad J := \int\limits_{\mathbb{S}^{N-1}} A(\hat{\xi}\cdot n, \hat{q}\cdot n)\hat{\xi}\wedge n\,\mathrm{d}\sigma(n)$$

are both scalar multiples of $\xi \wedge q$. Obviously, I and J inherit the anti-symmetry of their integrands, so

$$v^T I w = -w^T I v \quad \text{and} \quad v^T J w = -w^T J v \tag{10}$$

holds for arbitrary $v, w \in \mathbb{R}^N$. We shall now show that these products are actually zero whenever $w \in X^\perp$. Indeed, for $w \in X^\perp$,

$$Iw = \int\limits_{\mathbb{S}^{N-1}} A(\hat{q}\cdot\tilde{n}, \hat{\xi}\cdot\tilde{n})\hat{q}n^T w\,\mathrm{d}\sigma(n). \tag{11}$$

Perform a change of variables $n = R^T\tilde{n}$ with an orthogonal matrix R under the integral such that $Rx = x$ for $x \in X$ and $Ry = -y$ for $y \in X^\perp$. This change leaves the spherical measure invariant, and the integrand in (11) changes to

$$A\big(\hat{q}\cdot R^T\tilde{n}, \hat{\xi}\cdot R^T\tilde{n}\big)\hat{q}\big(R^T\tilde{n}\big)^T w = A(R\hat{q}\cdot\tilde{n}, R\hat{\xi}\cdot\tilde{n})\hat{q}\tilde{n}^T(Rw) = -A(\hat{q}\cdot\tilde{n}, \hat{\xi}\cdot\tilde{n})\hat{q}\tilde{n}^T w,$$

which shows $Iw = -Iw = 0$. Thus I is an anti-symmetric matrix that is trivial on $X^\perp$. But the space of anti-symmetric matrices on X is (at most) one-dimensional, and is spanned by $q \wedge \xi$. So I and — by a similar argument — J are scalar multiples of $q \wedge \xi$. To prove (9), consider another orthogonal change of variables $n = R\tilde{n}$, in which $R\hat{\xi} = \hat{q}$ and $R\hat{q} = \hat{\xi}$. We find

$$I = \int_{\mathbb{S}^{N-1}} A\big(\hat{q} \cdot R^T \tilde{n}, \hat{\xi} \cdot R^T \tilde{n}\big)\hat{q} \wedge \big(R^T \tilde{n}\big)\,d\sigma(\tilde{n}) = \int_{\mathbb{S}^{N-1}} A(R\hat{q} \cdot \tilde{n}, R\hat{\xi} \cdot \tilde{n})R^T\big((R\hat{q}) \wedge \tilde{n}\big)R\,d\sigma(\tilde{n}) = R^T J R.$$

Since $R^T(q \wedge \xi)R = \xi \wedge q = -q \wedge \xi$, it follows that $I = -J$. □

3. Estimate on the Fisher information

In order to arrive at (3), we employ (4) in the same way as done in [2]. We adopt the abbreviations $f' = f(v')$, $g'_* = g(v'_*)$, etc. By definition of Q_+, and since B is a Maxwellian kernel, the quotient $B(n \cdot \hat{q})f'g'_*/Q_+[f,g]$ defines — for every v — a probability density for integration w.r.t. $dv_*\,d\sigma(n)$. Now rewrite the quotient $\nabla Q_+[f,g]/Q_+[f,g]$ using (4) and apply Jensen's inequality to find

$$\left|\frac{\nabla Q_+(v)}{Q_+(v)}\right|^2 \leqslant \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \left|\frac{Y'_+ + \mathrm{P}_{n,q}[Y'_-]}{f'g'_*}\right|^2 \frac{B(n \cdot \hat{q})f'g'_*}{Q_+(v)}\,dv_*\,d\sigma(n).$$

Multiply this expression by $Q_+[f,g]$, integrate w.r.t. v, and change variables $(v, v_*) \leftrightarrow (v', v'_*)$ again to obtain an estimate on the Fisher information:

$$\mathcal{F}\big[Q_+[f,g]\big] \leqslant \frac{1}{2} \iiint_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} B(\hat{q} \cdot n)\frac{|Y_+ + \mathrm{P}_{q,n}[Y_-]|^2}{2fg_*}\,d\sigma(n)\,dv\,dv_*. \tag{12}$$

To finish the proof, two properties of the operators P are needed: the first is simply

$$\mathrm{P}_{q,n} + \mathrm{P}^T_{q,n} = 2(\hat{q} \cdot n)\mathbf{1}, \tag{13}$$

which follows from the anti-symmetry $(\hat{q} \wedge n)^T = -\hat{q} \wedge n$. The second is taken from [2, Lemmata 3 and 4]:

Lemma 3.1. *For arbitrary vectors $a, b \in \mathbb{R}^N \setminus \{0\}$ and $x \in \mathbb{R}^N$,*

$$\big|\mathrm{P}_{ab}[x]\big| \leqslant |x|. \tag{14}$$

Expand the square under the integral in (12), using (14) and (13):

$$\frac{|Y_+ + \mathrm{P}_{q,n}[Y_-]|^2}{2fg_*} = \frac{|Y_+|^2 + |\mathrm{P}_{q,n}[Y_-]|^2 + Y_+ \cdot (\mathrm{P}_{q,n} + \mathrm{P}^T_{q,n})[Y_-]}{2fg_*} \leqslant \frac{|Y_+|^2 + |Y_-|^2 + 2(\hat{q} \cdot n)Y_+ \cdot Y_-}{2fg_*}$$
$$= |\nabla\sqrt{f}\,|^2 g_* + f|\nabla\sqrt{g}_*|^2 + (\hat{q} \cdot n)\big\{|\nabla\sqrt{f}\,|^2 g_* - f|\nabla\sqrt{g}_*|^2\big\}.$$

To arrive at (3), insert this expansion into (12), use the Maxwell property (2), the definition of λ_B in (3), and the fact that f and g are probability densities.

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