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On resonances in disordered multi-particle systems

Sur les résonances dans un système à plusieurs particules en milieu désordonné

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A B S T R A C T

We assess the probability of resonances between sufficiently distant states $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{y} = (y_1, \ldots, y_N)$ in the configuration space of an N-particle disordered quantum system on the lattice $\mathbb{Z}^d$, $d \geqslant 1$. This includes the cases where the transition $\mathbf{x} \rightsquigarrow \mathbf{y}$ "shuffles" the particles in x, like the transition $(a, a, b) \rightsquigarrow (a, b, b)$ in a 3-particle system. In presence of a random external potential $V(\cdot, \omega)$ such pairs of configurations $(\mathbf{x}, \mathbf{y})$ give rise to strongly coupled random local Hamiltonians, so that eigenvalue concentration bounds are difficult to obtain (cf. Aizenman and Warzel (2009) [2]; Chulaevsky and Suhov (2009) [8]). This results in eigenfunction decay bounds weaker than expected. We show that more optimal bounds obtained so far only for 2-particle systems (Chulaevsky and Suhov (2008) [6]) can be extended to any $N > 2$.

R É S U M É

On établit une estimation de la probabilité de résonance entre deux états quantiques $\mathbf{x} = (x_1, \ldots, x_N)$ et $\mathbf{y} = (y_1, \ldots, y_N)$ dans $\mathbb{Z}^d$, $d \geqslant 1$, pour un système de $N \geqslant 3$ particules quantiques en milieu désordonné. Cette estimation généralise l'analogue de l'estimation de Wegner pour N particules, analogue démontrée précédemment dans (Chulaevsky et Suhov (2008, 2009) [6,7]). Ce résultat permet d'obtenir des estimations optimales de décroissance de fonctions propres pour les systèmes de $N > 2$ particules dans les milieux désordonnés, déjà démontrées dans (Chulaevsky et Suhov (2008) [6]) pour $N = 2$.

Version française abrégée

On utilisera les notations suivantes. Soit $V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ un champ aléatoire sur $\mathbb{Z}^d$. Étant donné une partie finie $Q \subset \mathbb{Z}^d$, on note $\xi_Q(\omega)$ la moyenne empirique

$$\xi_Q(\omega) = \frac{1}{|Q|} \sum_{x \in Q} V(x, \omega)$$

et η_x les « fluctuations » de V autour de la moyenne : $\eta_x = V(x, \omega) - \xi_Q(\omega)$, $x \in Q$. Ensuite, on note $\mathfrak{F}_{V,Q}$ la sigma-algèbre engendrée par les variables aléatoires $\{\eta_x, \ x \in Q; \ V(y, \cdot), y \notin Q\}$, et $F_\xi(\cdot|\mathfrak{F}_{V,Q})$ la fonction de répartition conditionnelle $F_\xi(s|\mathfrak{F}_{V,Q}) := \mathbb{P}\{\xi_Q \leqslant s \mid \mathfrak{F}_{V,Q}\}$. On supposera que le champ aléatoire V vérifie la condition suivante :

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CM(ν) : *Pour tout $R \geqslant 0$ il existe une fonction $\nu_R : \mathbb{R}_+ \to \mathbb{R}_+$ telle que $\nu_R(0) = 0$ et que, pour tout sous-ensemble fini $Q \subset \mathbb{Z}^d$ et tout $u \in \mathbb{Z}^d$, la fonction de répartition conditionnelle $F_\xi(\cdot|\mathfrak{F}_{V,Q})$ vérifie*

$$\forall t, s \in \mathbb{R}, \quad \operatorname{ess\,sup}\bigl|F_\xi(t|\mathfrak{F}_{V,Q}) - F_\xi(s|\mathfrak{F}_{V,Q})\bigr| \leqslant \nu_R\bigl(|t-s|\bigr). \tag{1}$$

On note $\mathbf{C}_L(\mathbf{u})$ le cube $\{\mathbf{y} \in \mathbb{Z}^d \mid |\mathbf{y} - \mathbf{u}| = L\} \subset \mathbb{Z}^d$ et d_S la « distance symétrisée »

$$d_S(\mathbf{x}, \mathbf{y}) = \min_{\pi \in \mathfrak{S}_N} |\mathbf{x} - \pi\mathbf{y}|,$$

où $\mathfrak{S}_N$ désigne le groupe des permutations agissant sur les composantes des vecteurs $\mathbf{y} = (y_1, \dots, y_N)$. Le résultat principal de cette Note est le théorème suivant :

Théorème 0.1. *Soit $V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ un champ aléatoire vérifiant la condition* **CM**(ν)*, et soient $\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')}$, $\mathbf{H}_{\mathbf{C}_{L''}(\mathbf{u}'')}$ les opérateurs de la forme* (3) *relatifs aux cubes $\mathbf{C}_{L'}(\mathbf{u}')$, $\mathbf{C}_{L''}(\mathbf{u}'') \subset \mathbb{Z}^{Nd}$, $0 < L', L'' \leqslant R$. Si $d_S(\mathbf{x}, \mathbf{y}) > 2(N+1)L$, alors, pour tout $s > 0$, on a*

$$\mathbb{P}\bigl\{\operatorname{dist}\bigl(\sigma(\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')}), \sigma(\mathbf{H}_{\mathbf{C}_{L''}(\mathbf{u}'')})\bigr) \leqslant s\bigr\} \leqslant \bigl|\mathbf{C}_{L'}\bigl(\mathbf{u}'\bigr)\bigr| \cdot \bigl|\mathbf{C}_{L''}\bigl(\mathbf{u}''\bigr)\bigr| \nu_R(2s).$$

1. Introduction

We study quantum systems in a disordered environment, usually referred to as Anderson-type models, due to the seminal paper by P. Anderson [3]. For nearly *fifty years* following its publication, the localization phenomena have been studied in the single-particle approximation. A rigorous mathematical analysis of localization in disordered quantum systems started approximately "Twenty Years After" the aforementioned Anderson's paper, when Goldsheid, Molchanov and Pastur [13] proved that the spectrum of a one-dimensional random Schrödinger operator with potential generated by a sufficiently regular Markov process is pure point (with probability one). Kunz and Souillard [15] proved a similar result for a lattice Anderson model in $\mathbb{Z}^1$.

Further progress in higher-dimensional Anderson-type models had been made approximately "Thirty Years After" the publication of Anderson's original paper, in the works by Fröhlich and Spencer [10], Fröhlich, Martinelli, Scoppola and Spencer [11] who developed an inductive procedure known as the Multi-Scale Analysis (MSA). The MSA procedure was reformulated later by von Dreifus and Klein [9]. An alternative approach — the Fractional Moment Method (FMM) — was developed by Aizenman and Molchanov [1]. A detailed discussion of Anderson models in Euclidean spaces can be found in the monograph [16].

To describe a system of $N > 1$ quantum particles in $\mathbb{Z}^d$, we consider the Hamiltonian $H_{V,U}(\omega)$ in the Hilbert space $\mathcal{H}_N := \ell^2(\mathbb{Z}^{Nd})$ of the form

$$H_{V,U} = \sum_{j=1}^{N} \bigl(\Delta^{(j)} + V(x_j, \omega)\bigr) + \mathbf{U}, \tag{2}$$

where

$$\Delta^{(j)}\Psi(x_1, \dots, x_N) = \sum_{\substack{y \in \mathbb{Z}^d \\ |y - x_j| = 1}} \Psi(x_1, \dots, x_j + y, \dots, x_N),$$

and $\mathbf{U}$ is the multiplication operator by a function $\mathbf{U}(\mathbf{x})$ which we assume bounded (this assumption can be relaxed so as to allow hard-core interactions). If both $\mathbf{U}$ and the random field $V(\cdot, \omega)$ are bounded, then $\mathbf{H}_{U,V} : \mathcal{H}_N \to \mathcal{H}_N$ is a (random) bounded self-adjoint operator. The symmetry of the function $\mathbf{U}$ is not required, and we do not assume $\mathbf{U}$ to be a "short-range" or rapidly decaying interaction.

We denote by $|\cdot|$ the max-norm $\|\cdot\|_\infty$. Given any finite cube $\mathbf{C}_L(\mathbf{u}) := \{\mathbf{x} \in \mathbb{Z}^{Nd} \mid |\mathbf{x} - \mathbf{u}| \leqslant L\}$, we will consider a finite-volume approximation of the Hamiltonian $\mathbf{H}$

$$\mathbf{H}_{\mathbf{C}_L(\mathbf{u})} = \mathbf{H}\restriction_{\ell^2(\mathbf{C}_L(\mathbf{u}))} \quad \text{with Dirichlet boundary conditions on } \partial\mathbf{C}_L(\mathbf{u}) \tag{3}$$

acting in the finite-dimensional Hilbert space $\ell^2(\mathbf{C}_L(\mathbf{u}))$. In this Note, we focus on eigenvalue concentration bounds, known as "Wegner-type bounds", due to the celebrated paper by Wegner [17]. In essence, one may call "Wegner-type bound" a probabilistic bound of the form

$$\mathbb{P}\bigl\{\operatorname{dist}\bigl(E, \sigma\bigl(H_\Lambda(\omega)\bigr)\bigr) \leqslant \epsilon\bigr\} \leqslant f\bigl(|\Lambda|, \epsilon\bigr). \tag{4}$$

In applications to the localization theory, for such a bound to be useful, the function f should decay not too slowly as $\epsilon \downarrow 0$. A Wegner bound alone, of the form (4), does not suffice in the context of multi-particle models. In the case where the marginal probability distribution function F_V of an IID random field V is analytic in a strip around the real axis (e.g., Gaussian, Cauchy), a multi-particle Wegner-type bound was proven in [5]. Kirsch [14] proved an analog of the finite-volume bound (4) for multi-particle systems under the assumption of bounded marginal density. However, it should be emphasized again that a one-volume Wegner-type bound (4), no matter how sharp, seems *insufficient* for the multi-particle MSA to work.

In [6] the following "two-volume" version of the Wegner bound was established for pairs of two-particle operators $\mathbf{H}_{\mathbf{C}_L(\mathbf{u})}$, $\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')}$ such that $L \geqslant L'$ and $\mathrm{dist}(\mathbf{C}_L(\mathbf{u}), \mathbf{C}_{L'}(\mathbf{u}')) > 8L$: if ν is the continuity modulus of the marginal distribution function F_V of the IID random field V, then for the (random) spectra $\sigma(\mathbf{H}_{\mathbf{C}_L(\mathbf{u})})$, $\sigma(\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')})$ of these operators one has

$$\mathbb{P}\big\{\mathrm{dist}\big(\sigma(\mathbf{H}_{\mathbf{C}_L(\mathbf{u})}), \sigma(\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')})\big) \leqslant \epsilon\big\} \leqslant (2L+1)^{2d}\big(2L'+1\big)^d \nu(2\epsilon). \tag{5}$$

Starting from $N = 3$, an additional difficulty appears in the analysis of pairs of spectra $\sigma(\mathbf{H}_{\mathbf{C}_L(\mathbf{u})})$, $\sigma(\mathbf{H}_{\mathbf{C}_L(\mathbf{u}')})$: no lower bound of the form $\mathrm{dist}(\mathbf{C}_L(\mathbf{u}), \mathbf{C}_L(\mathbf{u}')) > CL$ can guarantee the approach of [6] to work, no matter how large is the constant C. This gives rise to a significantly more sophisticated multi-particle MSA procedure and to less efficient eigenfunction decay bounds in the general case where $N \geqslant 3$. A similar difficulty arises also in the FMM approach to multi-particle systems, as was pointed out by Aizenman and Warzel in [2]. Physically speaking, it was difficult to rule out the possibility of "tunneling" between points $\mathbf{x}$ and $\mathbf{y}$ related by a "partial charge transfer" process, e.g., between points (a, a, b) and (a, b, b), corresponding to the quantum states $\mathbf{x}$ and $\mathbf{y}$:

state $\mathbf{x}$: 2 particles at the point a and 1 particle at b,

state $\mathbf{y}$: 1 particle at the point a and 2 particles at b.

Here, one particle is "transferred" from a to b. Observe that $|a - b|$ can be arbitrarily large.

In the present paper we address this problem and show that resonances between distant quantum states in the configuration space, related by partial charge transfer processes, are unlikely, and give explicit (*albeit not sharp*) *probabilistic estimates for such unlikely situations.*

2. Main result

Introduce the following notations. We denote by d_S the symmetrized norm-distance in $\mathbb{Z}^{Nd}$:

$$d_S(\mathbf{x}, \mathbf{y}) = \min_{\pi \in \mathfrak{S}_N} |\mathbf{x} - \pi\mathbf{y}|,$$

where $\mathfrak{S}_N$ is the group of permutations in $(\mathbb{Z}^d)^N$ acting on the components of the vectors $\mathbf{y} = (y_1, \ldots, y_N)$. Further, given a finite set $Q \subset \mathbb{Z}^d$, consider the sample mean $\xi_Q(\omega)$ of the random field V over Q,

$$\xi_Q(\omega) = \frac{1}{|Q|} \sum_{x \in Q} V(x, \omega),$$

and the "fluctuations" of V relative to the sample mean, $\eta_x = V(x, \omega) - \xi_Q(\omega)$, $x \in Q$. We denote by $\mathfrak{F}_{V,Q}$ the sigma-algebra generated by $\{\eta_x,\ x \in Q;\ V(y, \cdot),\ y \notin Q\}$, and by $F_\xi(\cdot|\mathfrak{F}_{V,Q})$ the conditional distribution function of ξ given $\mathfrak{F}_{V,Q}$:

$$F_\xi(s|\mathfrak{F}_{V,Q}) := \mathbb{P}\{\xi_Q \leqslant s \mid \mathfrak{F}_{V,Q}\}.$$

For a given $s \in \mathbb{R}$, $F_\xi(s|\mathfrak{F}_{V,Q})$ is, of course, a random variable, but we will write equalities or inequalities involving it, meaning that these relations hold true for $\mathbb{P}$-a.e. condition. It will be assumed that the random field V satisfies the following condition:

CM(ν): *For any* $R \geqslant 0$ *there exists a function* $\nu_R : \mathbb{R}_+ \to \mathbb{R}_+$ *vanishing at* 0 *and such that for any* $Q \subset \mathbb{Z}^d$ *with* $\mathrm{diam}(Q) \leqslant R$ *the conditional distribution function* $F_\xi(\cdot|\mathfrak{F}_{V,Q})$ *satisfies*

$$\forall t, s \in \mathbb{R}, \quad \operatorname{ess\,sup}\big[\big|F_\xi(t|\mathfrak{F}_{V,Q}) - F_\xi(s|\mathfrak{F}_{V,Q})\big|\big] \leqslant \nu_R\big(|t - s|\big). \tag{6}$$

Remarks.

(a) The condition **CM**(ν) is not automatically fulfilled even for IID random fields $V(x, \omega)$ with absolutely continuous marginal distribution. For example, the bound (6) is not uniform for $V(x; \omega)$ uniformly distributed in a finite interval $[a, b]$. Our method can, however, be adapted to IID potentials with compactly supported bounded probability density (see a forthcoming manuscript [12]).

(b) It is well-known that in the particular case where V is a Gaussian IID field of variance $\sigma^2 > 0$ the sample mean ξ_Q is a Gaussian variable *independent* of $\mathfrak{F}_{V,Q}$, of variance $\sigma^2/|Q|$.

The main result of this Note is the following:

Theorem 2.1. *Let $V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ be a random field satisfying* $\mathbf{CM}(\nu)$*, and $\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')}$, $\mathbf{H}_{\mathbf{C}_{L''}(\mathbf{u}'')}$ be the operators of the form* (3) *relative to the cubes $\mathbf{C}_{L'}(\mathbf{u}')$, $\mathbf{C}_{L''}(\mathbf{u}'') \subset \mathbb{Z}^{Nd}$ with $0 < L', L'' \leqslant R$. If $d_S(\mathbf{u}', \mathbf{u}'') > 2(N+1)R$, then for any $s > 0$ the following bound holds*:

$$\mathbb{P}\big\{\operatorname{dist}\big(\sigma(\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')}), \sigma(\mathbf{H}_{\mathbf{C}_{L''}(\mathbf{u}'')})\big) \leqslant s\big\} \leqslant \big|\mathbf{C}_{L'}\big(\mathbf{u}'\big)\big| \cdot \big|\mathbf{C}_{L''}\big(\mathbf{u}''\big)\big|\, \nu_R(2s).$$

3. Weak separability of particle configurations

Consider the lattice $(\mathbb{Z}^d)^N \cong \mathbb{Z}^{Nd}$, $N > 1$. We will use the notations $\mathbb{D} = \{\mathbf{x} \in \mathbb{Z}^{Nd}\colon\ \mathbf{x} = (x, \dots, x),\ \ x \in \mathbb{Z}^d\}$, $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$. Vectors $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{Z}^d \times \dots \times \mathbb{Z}^d$ will be identified with configurations of N distinguishable particles in $\mathbb{Z}^d$.

Definition 3.1 *(Subconfiguration).* Let $\mathbf{x} \in \mathbb{Z}^{Nd}$ be an N-particle configuration and consider a subset $\mathcal{J} \subset [\![1, N]\!]$ with $1 \leqslant |\mathcal{J}| = n < N$. A *subconfiguration* of $\mathbf{x}$ associated with $\mathcal{J}$ is a pair $(\mathbf{x}', \mathcal{J})$ where the vector $\mathbf{x}' \in \mathbb{Z}^{nd}$ has the components $x'_i = x_{j_i}$, $i \in [\![1, n]\!]$. Such a subconfiguration will be denoted as $\mathbf{x}_{\mathcal{J}}$. The complement of a subconfiguration $\mathbf{x}_{\mathcal{J}}$ is the subconfiguration $\mathbf{x}_{\mathcal{J}^c}$ associated with the complementary index subset $\mathcal{J}^c := [\![1, N]\!] \setminus \mathcal{J}$.

By a slight abuse of notations, we will identify a subconfiguration $\mathbf{x}_{\mathcal{J}} = (\mathbf{x}', \mathcal{J})$ with the vector $\mathbf{x}'$.

Definition 3.2.

(a) Let $N \geqslant 2$ and consider the set of all N-particle configurations $\mathbb{Z}^{Nd}$. For each $j \in [\![1, N]\!]$ the coordinate projection $\Pi_j : \mathbb{Z}^{Nd} \to \mathbb{Z}^d$ onto the coordinate space of the j-th particle is the mapping $\Pi_j : (x_1, \dots, x_N) \mapsto x_j$.
(b) The support $\Pi\mathbf{x}$ of a configuration $\mathbf{x} \in \mathbb{Z}^{nd}$, $n \geqslant 1$, is the set $\Pi\mathbf{x} := \bigcup_{j=1}^n \Pi_j\mathbf{x} = \{x_1, \dots, x_N\}$. Similarly, the support of a subconfiguration $\mathbf{x}_{\mathcal{J}}$ is defined by $\Pi\mathbf{x}_{\mathcal{J}} := \bigcup_{j\in\mathcal{J}}^n \Pi_j\mathbf{x}$.
(c) Given a non-empty subset $\mathcal{J} \subset [\![1, N]\!]$ with $|\mathcal{J}| = n$, the projection $\Pi_{\mathcal{J}} : \mathbb{Z}^{Nd} \to \mathbb{Z}^{nd}$ is defined as follows: $\Pi_{\mathcal{J}}\mathbf{x} = \Pi\mathbf{x}_{\mathcal{J}}$. For $\mathcal{J} = \varnothing$, we set, formally, $\Pi_\varnothing\mathbf{x} = \varnothing$. Finally, for each subset $\mathbf{C} \subset \mathbb{Z}^{Nd}$ its support $\Pi\mathbf{C}$ is defined by $\Pi\mathbf{C} := \bigcup_{j=1}^N \Pi_j\mathbf{C} \subset \mathbb{Z}^d$.

We will not associate with the empty subconfiguration $\mathbf{x}_\varnothing$ any object other than its support $\Pi\mathbf{x}_\varnothing = \varnothing \subset \mathbb{Z}^d$, so the above definitions and notations suffice for our purposes.

Particle configurations being associated with point subsets of $\mathbb{Z}^d$, one can introduce the distance between two arbitrary configurations $\mathbf{x}' \in \mathbb{Z}^{N'd}$, $\mathbf{x}'' \in \mathbb{Z}^{N''d}$, $N', N'' \geqslant 1$, as the Hausdorff distance between the respective subsets of $\mathbb{Z}^d$:

$$\operatorname{dist}_{\mathcal{H}}\big(\mathbf{x}', \mathbf{x}''\big) = \min_{i\in[\![1,N']\!]} \min_{j\in[\![1,N'']\!]} \big|x'_i - x''_j\big|.$$

Further, we call the canonical envelope of a bounded subset $\mathcal{X} \subset \mathbb{R}^{nd}$ the minimal parallelepiped $\mathcal{Q}$ such that $\mathcal{X} \subseteq \mathcal{Q}$. It will be denoted as $\mathcal{Q}(\mathcal{X})$. Next, for bounded sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^{nd}$ we set

$$d_{\mathcal{CH}}(\mathcal{X}, \mathcal{Y}) = d_{\mathcal{H}}\big(\mathcal{Q}(\mathcal{X}), \mathcal{Q}(\mathcal{Y})\big).$$

Definition 3.3 *(R-decoupling).* A configuration $\mathbf{x} \in \mathbb{Z}^{Nd}$, $N > 1$, is called *R-decoupled*, with $R > 0$, if it contains a pair of non-empty complementary subconfigurations $\mathbf{x}_{\mathcal{J}}, \mathbf{x}_{\mathcal{J}^c}$ with $d_{\mathcal{CH}}(\mathbf{x}_{\mathcal{J}}, \mathbf{x}_{\mathcal{J}^c}) > 2R$. The *decoupling width* of a configuration $\mathbf{x}$ is the quantity

$$D(\mathbf{x}) := \max_{\mathcal{J}, \mathcal{J}^c \subsetneq [\![1,N]\!]} d_{\mathcal{CH}}(\mathbf{x}_{\mathcal{J}}, \mathbf{x}_{\mathcal{J}^c}).$$

The *decoupling width* $D(\mathbf{C}_L(\mathbf{x}))$ of a cube $\mathbf{C}_L(\mathbf{x})$ is defined as follows:

$$D\big(\mathbf{C}_L(\mathbf{x})\big) := \max_{\mathcal{J}, \mathcal{J}^c \subsetneq [\![1,N]\!]} d_{\mathcal{CH}}\Big(\bigcup_{i\in\mathcal{J}} C_L(x_i), \bigcup_{j\in\mathcal{J}^c} C_L(x_j)\Big).$$

Definition 3.4 *(R-cluster).* An *R-cluster* of a configuration $\mathbf{x}$ is a subconfiguration $\mathbf{x}_{\mathcal{J}}$ which is not R-decoupled and is not contained in any other R-non-decoupled subconfiguration $\mathbf{x}_{\mathcal{J}'}$ with $|\mathcal{J}'| > |\mathcal{J}|$. The set of all R-clusters of a configuration $\mathbf{x}$ is denoted by $\Gamma(\mathbf{x}, R)$.

Lemma 3.5. *Fix a positive number R and integers $N > 1$, $d \geqslant 1$.*

(i) *Every configuration $\mathbf{x} \in \mathbb{Z}^{Nd}$ can be decomposed into a family $\Gamma(\mathbf{x}, R)$ of R-clusters $\Gamma_1, \dots, \Gamma_M$, with some $1 \leqslant M = M(\mathbf{x}) \leqslant N$, so that $(\mathcal{J}_{\Gamma_1}, \dots, \mathcal{J}_{\Gamma_M})$ is a partition of $[\![1, N]\!]$ and each particle x_j is contained in exactly one cluster $\Gamma = \Gamma_{\mathbf{x}}(x_j)$.*

(ii) *If $\Gamma', \Gamma'' \in \Gamma(\mathbf{x}, R)$ and $\Gamma' \neq \Gamma''$, then $d_{\mathcal{CH}}(\Gamma', \Gamma'') > 2R$.*
(iii) *The diameter of any cluster Γ is bounded by $2(N-1)R$, and so is the diameter of its envelope $\mathcal{Q}(\Gamma)$.*

Definition 3.6 *(Weak separability).* A cube $\mathbf{C}_L(\mathbf{x})$ is *weakly separable* from $\mathbf{C}_L(\mathbf{y})$ if there exists a parallelepiped $Q \subset \mathbb{Z}^d$ in 1-particle configuration space, of diameter $R \leqslant 2NL$, and subsets $\mathcal{J}_1, \mathcal{J}_2 \subset [\![1, N]\!]$ such that $|\mathcal{J}_1| > |\mathcal{J}_2|$ (possibly, with $\mathcal{J}_2 = \varnothing$) and

$$\Pi_{\mathcal{J}_1}\mathbf{C}_L(\mathbf{x}) \cup \Pi_{\mathcal{J}_2}\mathbf{C}_L(\mathbf{y}) \subseteq Q,$$
$$\Pi_{\mathcal{J}_2^c}\mathbf{C}_L(\mathbf{y}) \cap Q = \varnothing.$$

A pair $(\mathbf{C}_L(\mathbf{x}), \mathbf{C}_L(\mathbf{y}))$ is *weakly separable* if at least one of the cubes is weakly separable from the other.

The physical meaning of the "weak separability" is that in a certain region of the single-particle configuration space the presence of particles from configuration $\mathbf{x}$ is more important than that of the particles from $\mathbf{y}$. (This property is indeed *weaker* than "separability" used in [4–8].) As a result, certain local fluctuations of $V(u;\omega)$ have a stronger influence on $\mathbf{x}$ than on $\mathbf{y}$.

Lemma 3.7. *Two cubes $\mathbf{C}_L(\mathbf{x}), \mathbf{C}_L(\mathbf{y})$ with $d_S(\mathbf{x}, \mathbf{y}) > 2(N+1)L$ are weakly separable.*

The principal technical result of this Note is the following statement:

Lemma 3.8. *Let $V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ be a random field satisfying the condition $\mathbf{CM}(\xi, \nu)$. Let $\mathbf{u}', \mathbf{u}'' \in \mathbb{Z}^{Nd}$ be two weakly separable configurations and consider operators $\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')}, \mathbf{H}_{\mathbf{C}_{L''}(\mathbf{u}'')}$, with $L', L'' \leqslant R$. Then for any $s > 0$ the following bound holds for the random spectra $\sigma(\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')}), \sigma(\mathbf{H}_{\mathbf{C}_{L''}(\mathbf{u}'')})$ of these operators:*

$$\mathbb{P}\left\{\mathrm{dist}\left(\sigma(\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')}), \sigma(\mathbf{H}_{\mathbf{C}_{L''}(\mathbf{u}'')})\right) \leqslant s\right\} \leqslant \left|\mathbf{C}_{L'}\left(\mathbf{u}'\right)\right| \cdot \left|\mathbf{C}_{L''}\left(\mathbf{u}''\right)\right| \nu_R(2s).$$

Theorem 2.1 can be proven as follows. By assumption of the theorem, we have $d_S(\mathbf{u}', \mathbf{u}'') > 2(N+1)R$. Therefore, by Lemma 3.7, the cubes $\mathbf{C}_{L'}(\mathbf{u}')$ and $\mathbf{C}_{L''}(\mathbf{u}')$ are weakly separable. Now the assertion of the theorem follows from Lemma 3.8.

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