C. R. Acad. Sci. Paris, Ser. I 349 (2011) 1265–1267

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Differential Geometry

Lower bounds for the scalar curvatures of noncompact gradient Ricci solitons

Minorer des la courbures scalaires de solitons de Ricci gradient non compact

Bennett Chow [a], Peng Lu [b], Bo Yang [a]

[a] *Department of Mathematics, University of California, San Diego, La Jolla, CA 92093, United States*
[b] *Department of Mathematics, University of Oregon, Eugene, OR 97403, United States*

ARTICLE INFO

Article history:
Received 11 March 2011
Accepted 4 November 2011
Available online 16 November 2011

Presented by Jean-Pierre Demailly

ABSTRACT

We show that recent work of Ni and Wilking (in preparation) [11] yields the result that a noncompact nonflat Ricci shrinker has at most quadratic scalar curvature decay. The examples of noncompact Kähler–Ricci shrinkers by Feldman, Ilmanen, and Knopf (2003) [7] exhibit that this result is sharp. We also prove a similar result for certain noncompact steady gradient Ricci solitons.

RÉSUMÉ

Nous montrons que les travaux récents de Ni et Wilking (in preparation) [11] donne le résultat d'un non plate soliton contractant de type gradient non compact a tout au plus sa courbure scalaire avec décroissance quadratique. Les exemples de solitons de Kähler–Ricci contractant de type non compact par Feldman, Ilmanen, et Knopf (2003) [7] montre que ce résultat est optimales. Nous prouvons aussi un résultat similaire pour certains solitons de Ricci stable de type gradient non compact.

Let $(\mathcal{M}^n, g)$ be a complete Riemannian manifold, let f be a smooth function on $\mathcal{M}$, and let $\epsilon \in \mathbb{R}$. We say that the quadruple $(\mathcal{M}, g, f, \epsilon)$ is a complete gradient Ricci soliton if $R_{ij} + \nabla_i \nabla_j f + \frac{\epsilon}{2} g_{ij} = 0$. It is called shrinking (*Ricci shrinker* for short) if $\epsilon < 0$, steady if $\epsilon = 0$, and expanding if $\epsilon > 0$. Ricci solitons are self-similar solutions of the Ricci flow and often arise as blow-up limits of singular solutions of Ricci flow (see [9]). It is well known that $R + |\nabla f|^2 + \epsilon f$ is constant on Ricci solitons (see [9]).

It was proved by Bing-Long Chen [4] that $R \geqslant 0$ for Ricci shrinkers. If a Ricci shrinker is not isometric to Euclidean space, then $R > 0$ (see Stefano Pigola, Michele Rimoldi, and Alberto Setti [13] and Shijin Zhang [15]). Recently, Lei Ni and Burkhard Wilking [11] proved that on any noncompact nonflat Ricci shrinker and for any $\delta > 0$, there exists a constant $C_\delta > 0$ such that $R(x) \geqslant C_\delta d(x, O)^{-2-\delta}$ wherever $d(x, O)$ is sufficiently large. The purpose of this note is to observe the following version of their result and a similar result for certain noncompact steady gradient Ricci solitons.

Theorem 1. *Let $(\mathcal{M}^n, g, f, -1)$ be a complete noncompact nonflat Ricci shrinker with the potential function f normalized in the sense that $R + |\nabla f|^2 - f = 0$. Then for any given point $O \in \mathcal{M}$ there exists a constant $C_0 > 0$ such that $R(x) d(x, O)^2 \geqslant C_0^{-1}$ wherever $d(x, O) \geqslant C_0$. Consequently, the asymptotic scalar curvature ratio of g is positive.*

E-mail addresses: benchow@math.ucsd.edu (B. Chow), penglu@uoregon.edu (P. Lu), b5yang@math.ucsd.edu (B. Yang).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.11.004

Proof. Recall that Huai-Dong Cao and De-Tang Zhou [3] proved that on any complete shrinker there exists a positive constant C_1 such that f satisfies the estimate:

$$\frac{1}{4}\big[\big(d(x,O)-C_1\big)_+\big]^2 \leqslant f(x) \leqslant \frac{1}{4}\big(d(x,O)+2f(O)^{\frac{1}{2}}\big)^2, \tag{1}$$

where $c_+ \doteqdot \max(c,0)$ (see also Fu-Quan Fang, Jian-Wen Man, and Zhen-Lei Zhang [6] and, for an improvement, Robert Haslhofer and Reto Müller [10]). Define the f-Laplacian $\Delta_f \doteqdot \Delta - \nabla f\cdot\nabla$. We have $0 < R+|\nabla f|^2 = f = \frac{n}{2} - \Delta_f f$. Recall that (see [5] for example)

$$\Delta_f R = -2|\operatorname{Rc}|^2 + R. \tag{2}$$

Note that

$$\Delta_f\big(f^{-1}\big) = f^{-1} - f^{-2}\left(\frac{n}{2} - 2\frac{|\nabla f|^2}{f}\right), \tag{3}$$

$$\Delta_f\big(f^{-2}\big) = 2f^{-2} - f^{-3}\left(n - 6\frac{|\nabla f|^2}{f}\right). \tag{4}$$

Using (2) and (3), we compute for any $c>0$

$$\Delta_f\big(R - cf^{-1}\big) \leqslant R - cf^{-1} + cf^{-2}\left(\frac{n}{2} - 2\frac{|\nabla f|^2}{f}\right). \tag{5}$$

Define $\phi \doteqdot R - cf^{-1} - cnf^{-2}$. By (4) we obtain

$$\Delta_f\phi \leqslant \phi - cnf^{-3}\left(\frac{f}{2} - n\right) - cf^{-4}(2f+6n)|\nabla f|^2. \tag{6}$$

Choosing $c>0$ sufficiently small, we have $\phi>0$ inside $B(O,C_1+3n)$, where C_1 is as in (1). If $\inf_{\mathcal{M}-B(O,C_1+3n)}\phi \doteqdot -\delta < 0$, then by (1) there exists $\rho > C_1+3n$ such that $\phi > -\frac{\delta}{2}$ in $\mathcal{M}-B(O,\rho)$. Thus a negative minimum of ϕ is attained at some point x_0 outside of $B(O,C_1+3n)$. By the maximum principle, evaluating (6) at x_0 yields $\frac{f(x_0)}{2} - n \leqslant 0$. However, (1) implies that $f(x_0) \geqslant \frac{9n^2}{4}$, a contradiction. We conclude that $R \geqslant cf^{-1} + cnf^{-2}$ on $\mathcal{M}$. The theorem follows from (1). □

Remark. Mikhail Feldman, Tom Ilmanen, and Dan Knopf [7] constructed complete noncompact Kähler–Ricci shrinkers on the total spaces of k-th powers of tautological line bundles over the complex projective space $\mathbb{CP}^{n-1}$ for $0<k<n$. These examples, which have Euclidean volume growth and quadratic scalar curvature decay, show that Theorem 1 is sharp.

By a similar argument we prove the following result regarding steady gradient Ricci solitons. See [1,2,8,9], and [12] for some earlier works on the qualitative aspects of steady Ricci solitons.

Theorem 2. *Let $(\mathcal{M}^n, g, f, 0)$ be a complete steady gradient Ricci soliton with $R+|\nabla f|^2 = 1$. If $\lim_{x\to\infty} f(x) = -\infty$ and $f\leqslant 0$, then $R \geqslant \frac{1}{\sqrt{\frac{n}{2}}+2}e^f$.*

Proof. Note that on steady gradient Ricci solitons we have $\Delta_f f = -1$, $\Delta_f R = -2|\operatorname{Rc}|^2 \leqslant -\frac{2}{n}R^2$, and $\Delta_f(e^f) = -R\,e^f$. For $c\in\mathbb{R}$,

$$\Delta_f\big(R - ce^f\big) \leqslant -\frac{2}{n}R^2 + cR\,e^f \leqslant \frac{nc^2}{8}e^{2f}.$$

Using $\Delta_f(e^{2f}) = 2e^{2f}(1-2R)$, we compute for $b\in\mathbb{R}$ that

$$\Delta_f\big(R - ce^f - be^{2f}\big) \leqslant \left(\frac{nc^2}{8} - 2b + 4bR\right)e^{2f}. \tag{7}$$

Suppose $R - ce^f - be^{2f}$ is negative somewhere. Then, since $R\geqslant 0$ by [4] and $\lim_{x\to\infty} e^{f(x)} = 0$ by hypothesis, a negative minimum of $R - ce^f - be^{2f}$ is attained at some point. By (7) and the maximum principle, at such a point we have

$$0 \leqslant \frac{nc^2}{8} - 2b + 4bR < \frac{nc^2}{8} - 2b + 4b(c+b)$$

since $f\leqslant 0$. Given $c\in(0,\frac{1}{2}]$, the minimizing choice $b = \frac{1-2c}{4}$ yields $\frac{(1-2c)^2}{4} < \frac{nc^2}{8}$. We obtain a contradiction by choosing $c = \frac{1}{\sqrt{\frac{n}{2}}+2}$. □

Remark. Given a steady Ricci soliton $(\mathcal{M}^n, g, f, 0)$ with $R + |\nabla f|^2 = 1$ and $O \in \mathcal{M}$, since $|\nabla f| \leqslant 1$, we have $f(x) \geqslant f(O) - d(x, O)$ on $\mathcal{M}$. For the cigar soliton $(\mathbb{R}^2, \frac{4(\mathrm{d}x^2+\mathrm{d}y^2)}{1+x^2+y^2})$ we have $R = e^f$ assuming $\max_{x\in\mathbb{R}^2} f(x) = 0$. See [14] for an estimate for the potential functions of steady gradient Ricci solitons.

Acknowledgements

We would like to thank Lei Ni for informing us of his result with Burkhard Wilking.

References

[1] Simon Brendle, Uniqueness of gradient Ricci solitons, Mathematical Research Letters 18 (2011) 531–538.
[2] Huai-Dong Cao, Qiang Chen, On locally conformally flat gradient steady Ricci solitons, Transactions of the American Mathematical Society, in press.
[3] Huai-Dong Cao, De-Tang Zhou, On complete gradient shrinking Ricci solitons, Journal of Differential Geometry 85 (2010) 175–185.
[4] Bing-Long Chen, Strong uniqueness of the Ricci flow, Journal of Differential Geometry 82 (2009) 363–382.
[5] Manolo Eminenti, Gabriele La Nave, Carlo Mantegazza, Ricci solitons: the equation point of view, Manuscripta Mathematica 127 (2008) 345–367.
[6] Fu-Quan Fang, Jian-Wen Man, Zhen-Lei Zhang, Complete gradient shrinking Ricci solitons have finite topological type, Comptes Rendus Mathematique Academie des Sciences Paris 346 (2008) 653–656.
[7] Mikhail Feldman, Tom Ilmanen, Dan Knopf, Rotationally symmetric shrinking and expanding gradient Kähler–Ricci solitons, Journal of Differential Geometry 65 (2003) 169–209.
[8] Hongxin Guo, Area growth rate of the level surface of the potential function on the 3-dimensional steady Ricci soliton, Proceedings of the American Mathematical Society 137 (2009) 2093–2097.
[9] Richard S. Hamilton, The formation of singularities in the Ricci flow, in: Surveys in Differential Geometry, vol. II, International Press, Cambridge, MA, 1995, pp. 7–136.
[10] Robert Haslhofer, Reto Müller, A compactness theorem for complete Ricci shrinkers, Geometric and Functional Analysis 21 (5) (2011) 1091–1116.
[11] Lei Ni, Burkhard Wilking, in preparation.
[12] Ovidiu Munteanu, Natasa Sesum, On gradient Ricci solitons, arXiv:0910.1105, Journal of Geometric Analysis, in press.
[13] Stefano Pigola, Michele Rimoldi, Alberto G. Setti, Remarks on non-compact gradient Ricci solitons, Mathematische Zeitschrift 268 (3–4) (2011) 777–790, doi:10.1007/s00209-010-0695-4.
[14] Peng Wu, Remarks on gradient steady Ricci solitons, arXiv:1102.3018.
[15] Shijin Zhang, On a sharp volume estimate for gradient Ricci solitons with scalar curvature bounded below, Acta Mathematica Sinica 27 (5) (2011) 871–882.