



## Algebra

The  $D + E[\Gamma^*]$  construction from Prüfer domains and GCD-domainsConstruction  $D + E[\Gamma^*]$  d'anneaux de Prüfer et à pgcd

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## ABSTRACT

Let  $D \subsetneq E$  denote an extension of integral domains,  $\Gamma$  be a nonzero torsion-free grading monoid with  $\Gamma \cap -\Gamma = \{0\}$ ,  $\Gamma^* = \Gamma \setminus \{0\}$  and  $D + E[\Gamma^*] = \{f \in E[\Gamma] \mid f(0) \in D\}$ . In this paper, we give a necessary and sufficient criteria for  $D + E[\Gamma^*]$  to be a Prüfer domain or a GCD-domain.

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## R É S U M É

Soit  $D \subsetneq E$  une extension d'anneaux commutatifs intègres,  $\Gamma$  un monoïde commutatif simplifiable sans torsion non trivial tel que  $\Gamma \cap -\Gamma = \{0\}$ . On note  $\Gamma^* = \Gamma \setminus \{0\}$  et soit  $D + E[\Gamma^*] = \{f \in E[\Gamma] \mid f(0) \in D\}$ . Dans cette note, on donne des conditions nécessaires et suffisantes pour que  $D + E[\Gamma^*]$  soit un anneau de Prüfer ou un anneau à pgcd.

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## Version française abrégée

Soit  $D \subsetneq E$  une extension d'anneaux commutatifs intègres,  $K$  le corps des fractions de  $D$ ,  $\Gamma$  un monoïde commutatif simplifiable sans torsion non trivial tel que  $\Gamma \cap -\Gamma = \{0\}$ . On note  $\Gamma^* = \Gamma \setminus \{0\}$  et soit  $D + E[\Gamma^*] = \{f \in E[\Gamma] \mid f(0) \in D\}$ . Dans cet article on étudie la construction  $D + E[\Gamma^*]$  qui est un exemple important de produits fibrés d'anneaux. On montre que

- (1)  $D + E[\Gamma^*]$  est un anneau de Prüfer si et seulement si  $D$  est un anneau de Prüfer,  $\Gamma$  un sous-monoïde de Prüfer de  $\mathbb{Q}$  et  $E = K$ .
- (2)  $D + E[\Gamma^*]$  est un anneau à pgcd si et seulement si  $D$  est un anneau à pgcd,  $\Gamma$  un monoïde de valuation et  $E = D_S$  où  $S$  est une partie multiplicative de  $D$  qui satisfait des propriétés de divisibilité.

Comme corollaire, on montre que  $D + E[\Gamma^*]$  est un anneau de Bézout si et seulement si  $D$  est un anneau de Bézout,  $\Gamma$  un sous-monoïde de Prüfer de  $\mathbb{Q}$  et  $E = K$ .

## 1. Introduction

Let  $D \subsetneq E$  be an extension of integral domains,  $K$  be the quotient field of  $D$ ,  $\Gamma$  be a nonzero torsion-free grading monoid with  $\Gamma \cap -\Gamma = \{0\}$ ,  $\Gamma^* = \Gamma \setminus \{0\}$ ,  $E[\Gamma]$  be the semigroup ring of  $\Gamma$  over  $E$  and  $D + E[\Gamma^*] = \{f \in E[\Gamma] \mid f(0) \in D\}$  be the composite semigroup ring.

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Pullbacks have for many years been an important tool in the arsenal of commutative algebraists because of their use in producing many examples. In this paper, we devote to study the  $D + E[\Gamma^*]$  construction which is a nice example of them. The main purpose is to show the following:

- (1)  $D + E[\Gamma^*]$  is a Prüfer domain if and only if  $D$  is a Prüfer domain,  $\Gamma$  is a Prüfer submonoid of  $\mathbb{Q}$  and  $E = K$ .
- (2)  $D + E[\Gamma^*]$  is a GCD-domain if and only if  $D$  is a GCD-domain,  $\Gamma$  is a valuation semigroup and  $E = D_S$  for some splitting set  $S$  of  $D$ .

As a corollary, we prove that  $D + E[\Gamma^*]$  is a Bézout domain if and only if  $D$  is a Bézout domain,  $\Gamma$  is a Prüfer submonoid of  $\mathbb{Q}$  and  $E = K$ .

We briefly review some preliminaries. Recall that  $D$  is a *Prüfer domain* (resp., *Bézout domain*) if every nonzero finitely generated ideal of  $D$  is invertible (resp., principal); and  $D$  is a *GCD-domain* if  $aD \cap bD$  is principal for each  $0 \neq a, b \in D$ . It is well known that  $D$  is a Bézout domain if and only if  $D$  is a Prüfer GCD-domain. Recall that  $\Gamma$  is a *Prüfer submonoid* of  $\mathbb{Q}$  if it is the union of an ascending sequence of cyclic submonoids; and  $\Gamma$  is a *valuation semigroup* if for each  $\alpha, \beta \in \Gamma^*$ ,  $\alpha \in \beta + \Gamma$  or  $\beta \in \alpha + \Gamma$ .

Let  $\varphi$  be a generalized multiplicative system of  $D$ , i.e.,  $\varphi$  is a multiplicatively closed set of ideals of  $D$ . Then  $D_\varphi = \{x \in K \mid xA \subseteq D \text{ for some } A \in \varphi\}$ , which is called the  $\varphi$ -transform of  $D$  (or the *generalized transform* of  $D$  with respect to  $\varphi$ ). If  $I$  is an ideal of  $D$ , then  $I_\varphi = \{x \in K \mid xA \subseteq I \text{ for some } A \in \varphi\}$  is an ideal of  $D_\varphi$  containing  $I$  and  $ID_\varphi \subseteq I_\varphi$ ; but we may have  $ID_\varphi \neq I_\varphi$ . However,  $ID_\varphi = I_\varphi$  in case  $D_\varphi$  is  $D$ -flat [5, p. 185]. If  $E = D_\varphi$  for some generalized multiplicative system  $\varphi$  of  $D$  with  $D \subsetneq D_\varphi$ , then we write  $D + D_\varphi[\Gamma^*]$  instead of  $D + E[\Gamma^*]$ , i.e.,  $D + D_\varphi[\Gamma^*] = \{f \in D_\varphi[\Gamma^*] \mid f(0) \in D\}$ .

Notation and terminology used in this paper are standard as in [9–11]. The readers can also refer to [10] for the  $v$ -operations on integral domains and to [9] for semigroup rings.

## 2. Main results

We start this section with three lemmas that are required to give an attractive characterization of Prüfer domains via the  $D + E[\Gamma^*]$  construction.

**Lemma 2.1.** *The following assertions are equivalent:*

- (1) For all  $0 \neq d \in D$  and  $\alpha \in \Gamma^*$ ,  $(d, X^\alpha)$  is invertible in  $D + D_\varphi[\Gamma^*]$ .
- (2)  $D_\varphi = K$ .

**Proof.** (1)  $\Rightarrow$  (2) Let  $d \in D \setminus \{0\}$ . Then  $(d, X^\alpha)^{-1} = \frac{1}{d}(D + D_\varphi[\Gamma^*]) \cap \frac{1}{X^\alpha}(D + D_\varphi[\Gamma^*]) = (\frac{1}{d}D \cap D_\varphi) + D_\varphi[\Gamma^*]$  for each  $\alpha \in \Gamma^*$ . Since  $(d, X^\alpha)$  is invertible in  $D + D_\varphi[\Gamma^*]$ ,  $(d, X^\alpha)((\frac{1}{d}D \cap D_\varphi) + D_\varphi[\Gamma^*]) = D + D_\varphi[\Gamma^*]$ ; so  $d(\frac{1}{d}D \cap D_\varphi) = D$ . Hence  $\frac{1}{d} \in D_\varphi$ . Thus  $D_\varphi = K$ .

(2)  $\Rightarrow$  (1) Let  $0 \neq d \in D$  and  $\alpha \in \Gamma^*$ . Since  $D_\varphi = K$ , there exists an  $I \in \varphi$  such that  $\frac{1}{d}I \subseteq D$ ; so  $\frac{X^\alpha}{d}I \subseteq D[\Gamma^*]$ . Hence  $X^\alpha \in dD_\varphi[\Gamma^*] \subseteq d(D + D_\varphi[\Gamma^*])$ . Therefore  $(d, X^\alpha) = d(D + D_\varphi[\Gamma^*])$  is principal, and thus invertible.  $\square$

**Lemma 2.2.** *The following assertions hold for a nonzero fractional ideal  $I$  of  $D$ :*

- (1) If  $I$  is finitely generated, then  $(I(D + D_\varphi[\Gamma^*]))^{-1} = I^{-1}(D + D_\varphi[\Gamma^*])$ .
- (2)  $I$  is invertible in  $D$  if and only if  $I(D + D_\varphi[\Gamma^*])$  is invertible in  $D + D_\varphi[\Gamma^*]$ .

**Proof.** (1) The proof is similar to that of [7, Lemma 2.1].

(2) If  $I$  is invertible in  $D$ , then  $II^{-1} = D$ . Hence by (1),  $(I(D + D_\varphi[\Gamma^*]))(I(D + D_\varphi[\Gamma^*]))^{-1} = (I(D + D_\varphi[\Gamma^*]))(I^{-1}(D + D_\varphi[\Gamma^*])) = (II^{-1})(D + D_\varphi[\Gamma^*]) = D + D_\varphi[\Gamma^*]$ . Thus  $I(D + D_\varphi[\Gamma^*])$  is invertible in  $D + D_\varphi[\Gamma^*]$ . Conversely, if  $I(D + D_\varphi[\Gamma^*])$  is invertible in  $D + D_\varphi[\Gamma^*]$ , then we can find a finitely generated subideal  $J$  of  $I$  such that  $J(D + D_\varphi[\Gamma^*]) = I(D + D_\varphi[\Gamma^*])$ ; so  $(I(D + D_\varphi[\Gamma^*]))^{-1} = (J(D + D_\varphi[\Gamma^*]))^{-1} = J^{-1}(D + D_\varphi[\Gamma^*])$  by (1). Hence  $(IJ^{-1})(D + D_\varphi[\Gamma^*]) = (I(D + D_\varphi[\Gamma^*]))(J^{-1}(D + D_\varphi[\Gamma^*])) = D + D_\varphi[\Gamma^*]$ . Thus  $IJ^{-1} = (IJ^{-1})(D + D_\varphi[\Gamma^*]) \cap D = D$ , which means that  $I$  is invertible in  $D$ .  $\square$

**Lemma 2.3.** *The following assertions are equivalent:*

- (1)  $D + D_\varphi[\Gamma^*]$  is a Prüfer domain.
- (2)  $D$  is a Prüfer domain,  $\Gamma$  is a Prüfer submonoid of  $\mathbb{Q}$  and  $D_\varphi = K$ .

**Proof.** Let  $\phi : K[\Gamma] \rightarrow K = K[\Gamma]/K[\Gamma^*]$  be the canonical ring epimorphism, and consider the pullback diagram  $D + K[\Gamma^*]$  given by

$$\begin{array}{ccc}
 D + K[\Gamma^*] = \phi^{-1}(D) & \xrightarrow{\quad} & D \\
 \downarrow & & \downarrow \\
 K[\Gamma] & \xrightarrow{\phi} & K = K[\Gamma]/K[\Gamma^*].
 \end{array}$$

It is well known that  $D + K[\Gamma^*]$  is a Prüfer domain if and only if  $D$  and  $K[\Gamma]$  are Prüfer domains [3, Corollary 4.2(1)].

(1)  $\Rightarrow$  (2) Let  $I$  be a nonzero finitely generated ideal of  $D$ . Since  $D + D_\varphi[\Gamma^*]$  is a Prüfer domain,  $I(D + D_\varphi[\Gamma^*])$  is invertible in  $D + D_\varphi[\Gamma^*]$ ; so  $I$  is invertible in  $D$  by Lemma 2.2(2). Thus  $D$  is a Prüfer domain. Clearly,  $(d, X^\alpha)$  is invertible in  $D + D_\varphi[\Gamma^*]$  for any  $0 \neq d \in D$  and  $\alpha \in \Gamma^*$ . Hence  $D_\varphi = K$  by Lemma 2.1; so  $D + D_\varphi[\Gamma^*] = D + K[\Gamma^*]$ . Therefore  $K[\Gamma]$  is a Prüfer domain. Thus  $\Gamma$  is a Prüfer submonoid of  $\mathbb{Q}$  [9, Theorem 13.6].

(2)  $\Rightarrow$  (1) Since  $D_\varphi = K$ , we have  $D + D_\varphi[\Gamma^*] = D + K[\Gamma^*]$ . Also, since  $\Gamma$  is a Prüfer submonoid of  $\mathbb{Q}$ ,  $K[\Gamma]$  is a Prüfer domain [9, Theorem 13.6]. By the assumption,  $D$  is a Prüfer domain. Thus  $D + D_\varphi[\Gamma^*]$  is a Prüfer domain.  $\square$

Now, we are ready to give a complete characterization of Prüfer domains in terms of  $D + E[\Gamma^*]$ .

**Theorem 2.4.** *The following statements are equivalent:*

- (1)  $D + E[\Gamma^*]$  is a Prüfer domain.
- (2)  $D$  is a Prüfer domain,  $\Gamma$  is a Prüfer submonoid of  $\mathbb{Q}$  and  $E = K$ .

**Proof.** (1)  $\Rightarrow$  (2) If  $D + E[\Gamma^*]$  is a Prüfer domain, then  $D = D + E[\Gamma^*]/E[\Gamma^*]$  is also a Prüfer domain [10, Proposition 22.5]. Note that  $(D + E[\Gamma^*])_{D \setminus \{0\}} = K + E_{D \setminus \{0\}}[\Gamma^*]$ ; so  $K + E_{D \setminus \{0\}}[\Gamma^*]$  is also a Prüfer domain [6, Corollary 4.5]. Therefore  $E$  is an overring of  $D$  (cf. [12, Lemma 1.1]); so  $E$  is flat over  $D$  [10, Exercise 9, Section 40]. Hence  $E = D_\varphi$ , where  $\varphi = \{I \mid I \text{ is an ideal of } D \text{ such that } IE = E\}$  [4, Theorem 1.3] (or [5, Proposition 5.1]). Thus the result is an immediate consequence of Lemma 2.3.

(2)  $\Rightarrow$  (1) Lemma 2.3.  $\square$

Let  $v$  be the so-called  $v$ -operation on  $D$ , i.e., for any nonzero fractional ideal  $I$  of  $D$ ,  $I_v$  is the intersection of principal fractional ideals of  $D$  containing  $I$ . We say that a saturated multiplicative subset  $S$  of  $D$  is a *splitting set* of  $D$  if for each  $0 \neq d \in D$ , we can write  $d = sa$  for some  $s \in S$  and  $a \in N(S)$ , where  $N(S) = \{0 \neq x \in D \mid (x, t)_v = D \text{ for all } t \in S\}$ . In [7, Corollary 3.5], the authors showed that if  $D \subsetneq D_S$ , then  $D + D_S[\Gamma^*]$  is a GCD-domain if and only if  $D$  is a GCD-domain,  $\Gamma$  is a valuation semigroup and  $S$  is a splitting set of  $D$ . We extend this result to the general case.

**Theorem 2.5.** *The following statements are equivalent:*

- (1)  $D + E[\Gamma^*]$  is a GCD-domain.
- (2)  $D$  is a GCD-domain,  $\Gamma$  is a valuation semigroup and  $E = D_S$  for some splitting set  $S$  of  $D$ .

**Proof.** (1)  $\Rightarrow$  (2) Since  $D + E[\Gamma^*]$  is a GCD-domain,  $(D + E[\Gamma^*])_{D \setminus \{0\}}$  is also a GCD-domain; so  $E_{D \setminus \{0\}} = K$  [12, Lemma 1.1]. Hence  $E$  is an overring of  $D$ . Let  $S = \{d \in D \mid \frac{1}{d} \in E\}$ . Clearly,  $S$  is a nonempty saturated multiplicative subset of  $D$ . We claim that  $E = D_S$ . The containment  $D_S \subseteq E$  is obvious. For the reverse inclusion, let  $\frac{d}{s} \in E$ , where  $0 \neq d, s \in D$ . Since  $D + E[\Gamma^*]$  is a GCD-domain, we may assume that  $GCD(d, s) = 1$ , where  $GCD(d, s)$  stands for the greatest common divisor of  $d$  and  $s$  in the ring  $D + E[\Gamma^*]$ . Let  $\alpha \in \Gamma^*$ . Then  $dGCD(s, X^\alpha) = sGCD(d, \frac{d}{s}X^\alpha)$ ; so  $s \mid dGCD(s, X^\alpha)$ . Since  $GCD(d, s) = 1$ ,  $s \mid X^\alpha$  in  $D + E[\Gamma^*]$  [11, Exercise 7, Section 1.6]. Hence  $\frac{1}{s}X^\alpha \in D + E[\Gamma^*]$ , which indicates that  $\frac{1}{s} \in E$ ; so  $s \in S$ . Therefore  $\frac{d}{s} \in D_S$ , and hence  $E \subseteq D_S$ . Thus the claim is proved. Hence  $D + E[\Gamma^*] = D + D_S[\Gamma^*]$ . Note that  $D \subsetneq D_S$ . Thus  $D$  is a GCD-domain,  $S$  is a splitting set of  $D$  and  $\Gamma$  is a valuation semigroup [7, Corollary 3.5].

(2)  $\Rightarrow$  (1) This appears in [7, Corollary 3.5].  $\square$

The next corollary gives a necessary and sufficient condition for  $D + E[\Gamma^*]$  to be a Bézout domain.

**Corollary 2.6.** *The following statements are equivalent:*

- (1)  $D + E[\Gamma^*]$  is a Bézout domain.
- (2)  $D$  is a Bézout domain,  $\Gamma$  is a Prüfer submonoid of  $\mathbb{Q}$  and  $E = K$ .

**Proof.** Recall that  $D$  is a Bézout domain if and only if  $D$  is a Prüfer GCD-domain. We also note that  $\Gamma$  is a Prüfer submonoid of  $\mathbb{Q}$  if and only if  $\Gamma = H \cap \mathbb{Q}_0$ , where  $\mathbb{Q}_0$  is the semigroup of nonnegative rational numbers and  $H$  is a subgroup of  $\mathbb{Q}$  containing  $\mathbb{Z}$  [9, Theorem 13.5]; so a Prüfer submonoid of  $\mathbb{Q}$  is obviously a valuation semigroup, because  $\mathbb{Q}_0$  is a valuation

semigroup. It was shown that  $D \setminus \{0\}$  is a splitting set of  $D$  [1, Theorem 2.2]. Thus the result is an immediate consequence of Theorems 2.4 and 2.5.  $\square$

It is clear that  $\mathbb{N}_0$ , the semigroup of nonnegative integers, and  $\mathbb{Q}_0$  are Prüfer submonoids of  $\mathbb{Q}$ . Thus we have

**Corollary 2.7.** (Cf. [2, Corollary 2.6 and Theorem 3.6].) *Let  $S$  be a saturated multiplicative subset of  $D$  such that  $D \subsetneq D_S$  and  $\Gamma = \mathbb{N}_0$  or  $\mathbb{Q}_0$ . Then the following assertions hold:*

- (1)  $D + D_S[\Gamma^*]$  is a GCD-domain if and only if  $D$  is a GCD-domain and  $S$  is a splitting set of  $D$ .
- (2)  $D + D_S[\Gamma^*]$  is a Prüfer domain (resp., Bézout domain) if and only if  $D$  is a Prüfer domain (resp., Bézout domain) and  $D_S = K$ .

**Corollary 2.8.** (Cf. [8, Corollaries 1.3, 4.13 and 4.15].) *Let  $\Gamma = \mathbb{N}_0$  or  $\mathbb{Q}_0$ . Then  $D + K[\Gamma^*]$  is a Prüfer domain (resp., GCD-domain, Bézout domain) if and only if  $D$  is a Prüfer domain (resp., GCD-domain, Bézout domain).*

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