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Probability Theory

Harmonic moments and large deviations for a supercritical branching process in a random environment

Moments harmoniques et grandes déviations pour un processus de branchement sur-critique dans un environnement aléatoire

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A R T I C L E I N F O

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A B S T R A C T

Let (Z_n) be a supercritical branching process in a random environment ξ, and W be the limit of the normalized population size $Z_n/\mathbb{E}[Z_n|\xi]$. We show large and moderate deviation principles for the sequence $\log Z_n$ (with appropriate normalization) by finding an equivalence of the moments of Z_n and a criterion for the existence of harmonic moments of W.

R É S U M É

Soient (Z_n) un processus de branchement sur-critique dans un environnement aléatoire ξ, et W la limite de la population normalisée $Z_n/\mathbb{E}[Z_n|\xi]$. Nous montrons les principes de grande déviation et de déviation modérée pour la suite $\log Z_n$ en trouvant un équivalent des moments de Z_n et un critère pour l'existence des moments harmoniques de W.

1. Introduction and results

Let $\xi = (\xi_0, \xi_1, \xi_2, \ldots)$ be a sequence of independent and identically distributed (i.i.d.) random variables taking values in some space Θ, whose realization determines a sequence of probability generating functions

$$f_n(s) = f_{\xi_n}(s) = \sum_{i=0}^{\infty} p_i(\xi_n)s^i, \quad s \in [0,1], \qquad p_i(\xi_n) \geqslant 0, \qquad \sum_{i=0}^{\infty} p_i(\xi_n) = 1.$$

A branching process $(Z_n)_{n \geqslant 0}$ in the random environment ξ can be defined as follows:

$$Z_0 = 1, \qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \quad (n \geqslant 0),$$

where given the environment ξ, $X_{n,i}$ $(i = 1, 2, \ldots)$ are independent of each other and independent of Z_n, and have the same distribution determined by f_n.

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Let $(\Gamma, \mathbb{P}_\xi)$ be the probability space under which the process is defined when the environment ξ is given. As usual, $\mathbb{P}_\xi$ is called quenched law. The total probability space can be formulated as the product space $(\Gamma \times \Theta^{\mathbb{N}}, \mathbb{P})$, where $\mathbb{P} = \mathbb{P}_\xi \otimes \tau$, and τ is the law of the environment ξ. The total probability $\mathbb{P}$ is usually called annealed law. The quenched law $\mathbb{P}_\xi$ can be considered to be the conditional probability of the annealed law $\mathbb{P}$ given ξ. The expectation with respect to $\mathbb{P}_\xi$ (resp. $\mathbb{P}$) will be denoted by $\mathbb{E}_\xi$ (resp. $\mathbb{E}$).

For $n \geqslant 0$, define

$$m_n = m(\xi_n) = \sum_{i=0}^{\infty} i p_i(\xi_n), \qquad \Pi_0 = 1 \quad \text{and} \quad \Pi_n = m_0 \cdots m_{n-1} \quad \text{if } n \geqslant 1.$$

It is well known that the normalized population size $W_n = Z_n/\Pi_n$ is a nonnegative martingale under $\mathbb{P}_\xi$ (for each ξ) with respect to the filtration $\mathcal{F}_n = \sigma(\xi, X_{k,i}, 0 \leqslant k \leqslant n-1, i = 1, 2, \ldots)$, so that the limit $W = \lim W_n$ exists almost sure (a.s.) with $\mathbb{E}W \leqslant 1$.

We consider the supercritical case where $\mathbb{E}\log m_0 \in (0, \infty)$ and $\mathbb{E}\frac{Z_1}{m_0}\log^+ Z_1 < \infty$. For simplicity, we write $p_i := p_i(\xi_0)$ and assume that $p_0 = 0$ a.s., so that $W > 0$ and $Z_n \to \infty$ a.s.

It is known that $\frac{\log Z_n}{n} \to \mathbb{E}\log m_0$ a.s. (cf. e.g. [7]). We are interested in the convergence rate of the corresponding deviation probabilities. We shall show that $\log Z_n$ and $\log \Pi_n$ satisfy the same large and moderate deviation principles under suitable conditions.

We first consider large deviations. Let $\Lambda(t) = \log \mathbb{E} m_0^t < \infty$ for all $t \in \mathbb{R}$ and $\Lambda^*(x) = \sup_{t\in\mathbb{R}}\{xt - \Lambda(t)\}$ be the Fenchel–Legendre transform of Λ. We introduce the following assumption:

(H) *There exist constants $\delta > 0$ and $A > A_1 > 1$ such that a.s. $A_1 \leqslant m_0$, $m_0(1+\delta) \leqslant A^{1+\delta}$,*
where $m_0 = \sum_{i=0}^{\infty} i p_i(\xi_0)$ and $m_0(1+\delta) = \sum_{i=0}^{\infty} i^{1+\delta} p_i(\xi_0)$.

Notice that $m_0 = \mathbb{E}_\xi Z_1$, $m_0(1+\delta) = \mathbb{E}_\xi Z_1^{1+\delta}$ and that the above condition implies that $m_0 \leqslant A$ a.s.

The theorem below shows that $\log Z_n$ and $\log \Pi_n$ satisfy the same large deviation principle:

Theorem 1.1 *(Large deviation principle). Assume (H). If $\mathbb{E}Z_1^s < \infty$ for all $s > 1$ and $p_1 = 0$ a.s., then for any measurable subset B of $\mathbb{R}$,*

$$-\inf_{x\in B^o} \Lambda^*(x) \leqslant \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \in B\right) \leqslant \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \in B\right) \leqslant -\inf_{x\in \bar{B}} \Lambda^*(x),$$

where B^o denotes the interior of B, and $\bar{B}$ its closure.

Corollary 1.2. *Assume (H). If $\mathbb{E}Z_1^s < \infty$ for all $s > 1$ and $p_1 = 0$ a.s., then*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \leqslant x\right) = -\Lambda^*(x) \quad \textit{for } x < \mathbb{E}\log m_0,$$
$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log Z_n}{n} \geqslant x\right) = -\Lambda^*(x) \quad \textit{for } x > \mathbb{E}\log m_0.$$

This result was proved by Bansaye and Berestycki (2009, [1]) when (H) holds with $\delta = 1$. As shown in [1], if $\mathbb{P}(p_1 > 0) > 0$, the rate function for the large deviation is no longer Λ^*.

Notice that the Laplace transform of $\log Z_n$ is $\mathbb{E}e^{t\log Z_n} = \mathbb{E}Z_n^t$. Therefore, Theorem 1.1 is a consequence of the Gärtner–Ellis theorem (see [2, p. 52, Exercise 2.3.20]) and Theorem 1.3 below:

Theorem 1.3 *(Moments of Z_n). Let $t \in \mathbb{R}$. Suppose that one of the following conditions is satisfied:*

(i) *$t \in (0, 1]$ and $\mathbb{E}m_0^{t-1} Z_1 \log^+ Z_1 < \infty$;*
(ii) *$t > 1$ and $\mathbb{E}Z_1^t < \infty$;*
(iii) *$t < 0$, $\mathbb{E}p_1 < \mathbb{E}m_0^t$, $\|p_1\|_\infty :=$ esssup $p_1 < 1$ and (H) holds.*

Then as $n \to \infty$, $\mathbb{E}Z_n^t/(\mathbb{E}m_0^t)^n \to C(t)$ for some constant $C(t) \in (0, \infty)$.

For $t < 0$, Theorem 1.3 is an extension of a result of Ney and Vidyashankar [6] on the Galton–Watson process.

A key step in the proof of Theorem 1.3 is the study of the moments of W. For the moments of positive orders, Guivarc'h and Liu [3] showed that for $p > 1$, $\mathbb{E}W^p \in (0, \infty)$ if and only if $\mathbb{E}(Z_1/m_0)^p < \infty$ & $\mathbb{E}m_0^{1-p} < 1$. For the moments of negative orders, we have the following criterion:

Theorem 1.4 *(Harmonic moments of W). Assume (H). Then there exists a constant $a > 0$ such that $\mathbb{E}W^{-a} < \infty$. If additionally $\|p_1\|_\infty < 1$, then for $a > 0$, $\mathbb{E}W^{-a} < \infty$ if and only if $\mathbb{E}p_1 m_0^a < 1$.*

Theorem 1.4 reveals that under certain conditions, the number $a_0 > 0$ satisfying $\mathbb{E}p_1 m_0^{a_0} = 1$ is the critical value for the existence of the annealed harmonic moments $\mathbb{E}W^{-a}(a > 0)$. Hambly [4] proved that under an assumption similar to (H), the number $\alpha_0 := -\frac{\mathbb{E}\log p_1}{\mathbb{E}\log m_0}$ is the critical value for the a.s. existence of the quenched moments $\mathbb{E}_\xi W^{-a}(a > 0)$. By Jensen's inequality, we see that $a_0 \leqslant \alpha_0$.

We then show that $\log Z_n$ and $\log \Pi_n$ also satisfy the same moderate deviation principle.

Theorem 1.5 *(Moderate deviation principle). Assume (H) and let $\sigma^2 = \operatorname{var}(\log m_0) \in (0, \infty)$. Let (a_n) be a sequence of positive numbers satisfying $\frac{a_n}{n} \to 0$ and $\frac{a_n}{\sqrt{n}} \to \infty$. Then for any measurable subset B of $\mathbb{R}$,*

$$-\inf_{x \in B^o} \frac{x^2}{2\sigma^2} \leqslant \liminf_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \in B\right)$$
$$\leqslant \limsup_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} \in B\right) \leqslant -\inf_{x\in\bar{B}} \frac{x^2}{2\sigma^2},$$

where B^o denotes the interior of B, and $\bar{B}$ its closure.

2. Sketch of proofs

To prove the results about the harmonic moments of W, we consider the Laplace transform of W. Set $\phi_\xi(t) = \mathbb{E}_\xi e^{-tW}$ and $\phi(t) = \mathbb{E}\phi_\xi(t)$ for $t > 0$. The following lemma gives uniform upper bounds for $\phi_\xi(t)$:

Lemma 2.1. *Assume (H). Then there exist constants $\beta \in (0, 1)$ and $K > 0$ such that a.s. $\phi_\xi(t) \leqslant \beta$ for all $t \geqslant 1/K$. If additionally $\|p_1\|_\infty < 1$, then for some constants $a > 0$ and $C > 0$, we have a.s. $\phi_\xi(t) \leqslant Ct^{-a}$ for all $t \geqslant 1/K$.*

Proof. We obtain the upper bound β by an argument similar to [4, Proof of Lemma 3.1]. For the special case where $\|p_1\|_\infty < 1$, notice that $\phi_\xi(t)$ satisfies the functional equation

$$\phi_\xi(t) = f_0\big(\phi_{T\xi}(t/m_0)\big), \tag{1}$$

where $T^n\xi = (\xi_n, \xi_{n+1}, \ldots)$ if $\xi = (\xi_0, \xi_1, \ldots)$ and $n \geqslant 0$. By iteration, we have a.s.

$$\phi_\xi(t) \leqslant \phi_{T^n\xi}(t/\Pi_n) \prod_{j=0}^{n-1} \big(p_1(\xi_j) + \big(1 - p_1(\xi_j)\big)\phi_{T^n\xi}(t/\Pi_n)\big).$$

Since $\phi_{T^n\xi}(t/\Pi_n) \leqslant \beta$ for $t \geqslant A^n/K$, it follows that a.s. $\phi_\xi(t) \leqslant \beta\alpha^n$ for $t \geqslant A^n/K$, where $\alpha = \|p_1\|_\infty + (1 - \|p_1\|_\infty)\beta \in (0, 1)$. For $t \geqslant 1/K$, taking $n_0 = [\frac{\log(Kt)}{\log A}]$ yields the upper bound Ct^{-a} for suitable $a > 0$ and $C > 0$. □

Proof of Theorem 1.4. We first consider the special case where $\|p_1\|_\infty < 1$. For the necessity, notice that $W = \frac{1}{m_0}\sum_{i=1}^{Z_1} W_i^{(1)}$, where given ξ, $(W_i^{(1)})_{i\geqslant 1}$ are (conditionally) independent, each has the distribution $\mathbb{P}_\xi(W_i^{(1)} \in \cdot) = \mathbb{P}_{T\xi}(W \in \cdot)$. Since $\mathbb{P}(Z_1 \geqslant 2) > 0$, we have

$$\mathbb{E}W^{-a} > \mathbb{E}m_0^a\big(W_1^{(1)}\big)^{-a}\mathbf{1}_{\{Z_1=1\}} = \mathbb{E}p_1 m_0^a \mathbb{E}W^{-a}.$$

Thus $\mathbb{E}p_1 m_0^a < 1$. For the sufficiency, the upper bound Ct^{-a} in Lemma 2.1 implies that $\forall \varepsilon > 0$, there exists a constant $t_\varepsilon > 0$ such that a.s. $\phi_\xi(t) \leqslant \varepsilon$ for $t \geqslant t_\varepsilon$. Therefore, by (1), we have $\phi_\xi(t) \leqslant (p_1 + (1 - p_1)\varepsilon)\phi_{T\xi}(t/m_0)$ for $t \geqslant At_\varepsilon$. Taking the expectation gives

$$\phi(t) \leqslant \mathbb{E}\big(p_1 + (1 - p_1)\varepsilon\big)\phi\left(\frac{t}{m_0}\right) = p_\varepsilon \mathbb{E}\phi(\tilde{A}_\varepsilon t),$$

where $p_\varepsilon = \mathbb{E}(p_1 + (1 - p_1)\varepsilon) < 1$ and $\tilde{A}_\varepsilon$ is a positive random variable whose distribution is determined by $\mathbb{E}g(\tilde{A}_\varepsilon) = \frac{1}{p_\varepsilon}\mathbb{E}(p_1 + (1 - p_1)\varepsilon)g(\frac{1}{m_0})$ for all bounded and measurable function g. Since $\mathbb{E}p_1 m_0^a < 1$, we can take $a_1 > a$ and $\varepsilon > 0$ small enough such that $p_\varepsilon \mathbb{E}\tilde{A}_\varepsilon^{-a_1} < 1$. Then by Lemma 3.2 of Liu (2001, [5]), $\phi(t) = O(t^{-a_1})(t \to \infty)$. Therefore $\mathbb{E}W^{-a} < \infty$ (cf. e.g. [5, Lemma 3.3]).

Now consider the general case without the assumption $\|p_1\|_\infty < 1$. Notice that by Lemma 2.1, a.s. $\phi_\xi(t) \leqslant \beta$ for all $t \geqslant t_\beta = 1/K$. It suffices to repeat the proof of sufficiency above with β in place of ε. □

Proof of Theorem 1.3. Denote the distribution of ξ_0 by τ_0. Fix $t \in \mathbb{R}$ and define a new distribution $\tilde{\tau}_0$ as $\tilde{\tau}_0(dx) = m(x)^t \tau_0(dx)/\mathbb{E}m_0^t$, where $m(x) = \mathbb{E}[Z_1|\xi_0 = x] = \sum_{i=0}^{\infty} ip_i(x)$. Consider the new BPRE whose environment distribution is $\tilde{\tau} = \tilde{\tau}_0^{\otimes\mathbb{N}}$ instead of $\tau = \tau_0^{\otimes\mathbb{N}}$. The corresponding total probability and expectation are denoted by $\tilde{\mathbb{P}} = \mathbb{P}_\xi \otimes \tilde{\tau}$ and $\tilde{\mathbb{E}}$. Then $\mathbb{E}Z_n^t/(\mathbb{E}m_0^t)^n = \tilde{\mathbb{E}}W_n^t$. We distinguish three cases: $t \in (0,1)$, $t > 1$ and $t < 0$. For each case, under the given moment conditions, $\tilde{\mathbb{E}}W_n^t \to \tilde{\mathbb{E}}W^t \in (0,\infty)$. □

Proof of Theorem 1.5. Let

$$\Lambda_n(t) = \log \mathbb{E}\exp\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{a_n} t\right) \quad \text{and} \quad \Gamma_n(t) = \log \mathbb{E}\exp\left(\frac{\log \Pi_n - n\mathbb{E}\log m_0}{a_n} t\right).$$

By the classic moderate deviation principle, $\frac{n}{a_n^2}\Gamma_n(\frac{a_n^2}{n}t) \to \frac{1}{2}\sigma^2 t^2$. Applying Jensen's inequality and Hölder's inequality, we can prove that $\Lambda_n(\frac{a_n^2}{n}t)/\Gamma_n(\frac{a_n^2}{n}t) \to 1$ for all $t \neq 0$, so that $\frac{n}{a_n^2}\Lambda_n(\frac{a_n^2}{n}t) \to \frac{1}{2}\sigma^2 t^2$. This together with the Gärtner–Ellis theorem [2, p. 52, Exercise 2.3.20] implies the desired result. □

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References

[1] V. Bansaye, J. Berestycki, Large deviations for branching processes in random environment, Markov Process. Related Fields 15 (2009) 493–524.
[2] A. Dembo, O. Zeitouni, Large Deviations Techniques and Applications, Springer, New York, 1998.
[3] Y. Guivarc'h, Q. Liu, Propriétés asymptotiques des processus de branchement en environnement aléatoire, C. R. Acad. Sci. Paris, Ser. I 332 (2001) 339–344.
[4] B. Hambly, On the limit distribution of a supercritical branching process in a random environment, J. Appl. Probab. 29 (1992) 499–518.
[5] Q. Liu, Asymptotic properties and absolute continuity of laws stable by random weighted mean, Stochastic Process. Appl. 95 (2001) 83–107.
[6] P.E. Ney, A.N. Vidyashankar, Harmonic moments and large deviation rates for supercritical branching process, Ann. Appl. Probab. 13 (2003) 475–489.
[7] D. Tanny, Limit theorems for branching processes in a random environment, Ann. Probab. 5 (1977) 100–116.