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Partial Differential Equations

The Fujita phenomenon in exterior domains under the Robin boundary conditions

Le phénomène de Fujita dans un domaine extérieur sous les conditions au bord de Robin

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ABSTRACT

We use comparison methods, as in the case of the dynamical boundary conditions, to prove that the well-known Fujita phenomenon remains true in an exterior domain of $\mathbb{R}^N$ under the Robin boundary conditions.

RÉSUMÉ

Nous utilisons des méthodes de comparaison, comme dans le cas des conditions au bord dynamiques, pour démontrer que le phénomène de Fujita est également vérifié dans un domaine extérieur sous les conditions au bord de Robin.

1. Introduction

In an exterior domain Ω of $\mathbb{R}^N$ which boundary $\partial\Omega$ is of class $\mathcal{C}^2$, we consider the following parabolic problem

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \overline{\Omega} \text{ for } t > 0, \\ \partial_\nu u + \alpha u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = \varphi & \text{in } \overline{\Omega}, \end{cases} \tag{1}$$

where $p > 1$ is a real number, α a non-negative continuous function on $\partial\Omega \times \mathbb{R}^+$ and φ a continuous function in $\overline{\Omega}$. We aim to prove that the well-known Fujita phenomenon (see Refs. [2,5,6,8] and [10]) remains true under the Robin boundary conditions. Throughout, we shall assume that α is non-negative

$$\alpha \geqslant 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{2}$$

and, in order to deal with classical solutions, we need some regularity on α

$$\alpha \in \mathcal{C}\big(\partial\Omega \times \mathbb{R}^+\big). \tag{3}$$

To use the comparison method (see the truncation procedure described in Section 2 of [8]), we need

$$\varphi \in \mathcal{C}(\overline{\Omega}),\ 0 < \|\varphi\|_\infty < \infty,\ \varphi \geqslant 0, \qquad \lim_{\|x\|_2 \to \infty} \varphi(x) = 0. \tag{4}$$

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From [4], the unique classical solution of Problem (1) $u \in \mathcal{C}(\overline{\Omega} \times [0,T)) \cap \mathcal{C}^{2,1}(\overline{\Omega} \times (0,T))$ with maximal existence time $T > 0$ satisfies $T = \infty$ and u is called a global solution *or* $T < \infty$ and u blows up in finite time in the L^∞-norm. In the second case, we have

$$\lim_{t \nearrow T} \sup_{x \in \overline{\Omega}} u(x,t) = \infty.$$

2. Main results

Using the comparison method described in [8], we just need to find some appropriate sub-solutions to prove the blow-up case ($1 < p < 1 + 2/N$) and some adequate super-solution to obtain the global existence case ($p > 1 + 2/N$).

Theorem 2.1. *Suppose that conditions* (2), (3) *and* (4) *are fulfilled. Then all non-trivial positive solutions of Problem* (1) *blow up in finite time for $p \in (1, 1 + 2/N)$. Moreover, if $N \geqslant 3$, blow up also occurs for $p = 1 + 2/N$.*

Proof. Ab absurdo, suppose that there exists α and a non-trivial φ satisfying the hypotheses above, and such that the solution u of Problem (1) with these parameters is global. By the truncation method and with the comparison principle from [3], we construct a global solution v of the following Dirichlet problem

$$\begin{cases} \partial_t v = \Delta v + v^p & \text{in } \overline{\Omega} \times (0, +\infty), \\ v = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ v(\cdot, 0) = \varphi_0 & \text{in } \overline{\Omega}, \end{cases}$$

where $0 \leqslant \varphi_0 \leqslant \varphi$ and $\varphi_0 \in \mathcal{C}_0(\overline{\Omega})$. Thanks to the results of Bandle and Levine results [2] (see [1] for the one-dimensional case), we obtain a contradiction because the solution v must blow-up in finite time. If $N \geqslant 3$ and $p = 1 + 2/N$, the contradiction holds with Mochizuki and Suzuki's results [7] and [9]. □

Theorem 2.2. *Under hypotheses* (2), (3) *and* (4), *for $N \geqslant 3$ and*

$$p > 1 + \frac{2}{N},$$

Problem (1) *admits global non-trivial positive solutions for sufficiently small initial data φ.*

Proof. Consider the classical solution of the following Neumann problem

$$\begin{cases} \partial_t v = \Delta v + v^p & \text{in } \overline{\Omega} \text{ for } t > 0, \\ \partial_\nu v = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ v(\cdot, 0) = \varphi & \text{in } \overline{\Omega}. \end{cases}$$

By hypothesis (2), $\alpha \geqslant 0$ and by definition of v, we obtain

$$\partial_\nu v + \alpha v = \alpha v \geqslant 0 \quad \text{on } \partial\Omega \text{ for } t > 0.$$

Thus, v is clearly a super-solution of Problem (1). The comparison method previously used leads to $0 \leqslant u \leqslant v$ in Ω and for $t > 0$, where u is a solution of (1) with the parameters α and φ. If the initial data φ is small enough, the function v is global, see Levine and Zhang's results from [6]. Hence, the solution u cannot blow up in finite time, and it is a global solution. □

The global existence theorem is true only for high dimensions ($N \geqslant 3$) because Levine and Zhang's results are no more satisfied in dimension 2. For this special case, we impose a restriction on the coefficient α: suppose that there exists a positive constant $c > 0$ such that

$$\alpha \geqslant c \quad \text{on } \partial\Omega \times \mathbb{R}^+. \tag{5}$$

We use another super-solution, already used before by Bandle and Levine's in [1].

Theorem 2.3. *Let α be a coefficient satisfying* (3) *and* (5), *φ an initial data with* (4). *For*

$$p > 1 + \frac{2}{N},$$

Problem (1) *admits global positive solutions for sufficiently small initial data φ.*

Proof. Let U be the function defined in $\overline{\Omega} \times [0, \infty)$ by

$$U(x,t) = A(t+t_0)^{-\mu} \exp\left(-\frac{\|x\|_2^2}{4(t+t_0)}\right),$$

where $\mu = 1/(p-1)$, $t_0 > 0$ and $A > 0$ will be chosen below. A simple calculus of the derivatives leads to

$$\partial_t U(x,t) - \Delta U(x,t) \geqslant \frac{N-2\mu}{2(t+t_0)} U(x,t) \quad \text{in } \overline{\Omega} \times [0,\infty).$$

Since of $N - 2\mu > 0$ by definition of μ, and with $U^{p-1} \leqslant A^{p-1}(t+t_0)^{-1}$, we just need to choose $A > 0$ sufficiently small to obtain

$$\partial_t U - \Delta U \geqslant U^p \quad \text{in } \overline{\Omega} \times [0,\infty).$$

On the boundary $\partial\Omega$, hypothesis (5) gives

$$\begin{aligned} \partial_\nu U(x,t) + \alpha U(x,t) &\geqslant \left(\frac{-x\cdot\nu(x)}{2(t+t_0)} + \alpha(x,t)\right) U(x,t) \\ &\geqslant \left(\frac{-x\cdot\nu(x)}{2(t+t_0)} + c\right) U(x,t). \end{aligned} \tag{6}$$

As $\partial\Omega$ is compact, and with t_0 sufficiently big, this last term is non-negative. Thus, if $0 \leqslant \varphi \leqslant U(\cdot,0)$ in $\overline{\Omega}$, the function U is a global bounded super-solution. □

In dimension one, an exterior domain does not exist, but we can consider the case $\Omega = \mathbb{R} \setminus [a,b]$ with $a < b$ two real numbers. We obtain:

Theorem 2.4. *Let α be a coefficient satisfying* (3) *and φ an initial data with* (4)*. For*

$$p > 3,$$

Problem (1) *admits global positive solutions for sufficiently small initial data φ.*

Proof. Use the same super-solution U as in the previous proof. Modifications only appear in Eq. (6). Up to a translation, and thanks to the symmetry of the problem, we can treat only the case $\Omega = (0,\infty)$. On the boundary, we have $x = 0$ and $\partial_\nu U = 0$. Thus Eq. (6) is satisfied for all $\alpha \geqslant 0$. □

Remark 1. As in Bandle and Levine's results [1], we can consider a more general non-linear reaction term of the form $t^q \|x\|_2^s u^p$. In this case, we prove that the Fujita exponent is $1 + (2+2q+s)/N$. If $p < 1 + (2+2q+s)/N$, then nontrivial positive solutions of

$$\begin{cases} \partial_t u = \Delta u + t^q \|x\|_2^s u^p & \text{in } \overline{\Omega} \times (0,+\infty), \\ \partial_\nu u + \alpha u = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(\cdot,0) = \varphi & \text{in } \overline{\Omega}, \end{cases} \tag{7}$$

blow up in finite time. If $p > 1 + (2+2q+s)/N$, then there exist non-trivial global positive solutions if φ is small enough.

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