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The Thue–Morse–Pascal double sequence and similar structures

Suite double de Thue–Morse–Pascal et structures semblables

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A R T I C L E   I N F O

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A B S T R A C T

If a recurrent two-dimensional sequence with initial conditions defined by linear substitution and a two-dimensional sequence that is generated by planar substitution are identical over a sufficiently large initial square, then they will coincide over all. After proving this general principle, we apply it to some concrete examples. One of them, the Thue–Morse–Pascal two-dimensional sequence, is defined by two copies of the Prouhet–Thue–Morse sequence as pair of initial conditions and by the Pascal Triangle Addition modulo 2 as rule of recurrence. As it follows, the Thue–Morse–Pascal two-dimensional sequence is the result of 15 substitution rules, each of them consisting of the substitution of some 4×4 matrix with an 8×8 matrix.

R É S U M É

Si une suite bidimensionnelle récurrente avec conditions initiales définies par substitution linéaire et une suite bidimensionnelle engendrée par substitution plane sont identiques sur un carré initial assez grand, alors elles coïncident partout. Après avoir démontré ce principe on l'applique à quelques exemples concrets. L'un d'entre-eux est la suite bidimensionnelle de Thue–Morse–Pascal définie par deux exemplaires de la suite de Prouhet–Thue–Morse comme couple de conditions initiales et l'addition du triangle de Pascal modulo 2 comme règle de récurrence. Il s'ensuit que la suite bidimensionnelle de Thue–Morse–Pascal est le résultat de 15 règles de substitution, chacune d'entre-elles consistant de la substitution d'une certaine matrice 4×4 avec une matrice 8×8.

1. Introduction

This note reveals a new touching point between recurrence and substitution. Both notions occur in a field of interdisciplinary investigations unifying very heterogeneous motivations and techniques. The recurrence – although a very classical task – is more and more present in studies concerning cellular automata, see [26,12,7] or the monograph [27]. Substitutions occur in various contexts such as automatic sequences [11,1,2], aperiodic tilings [25,21,13,6,5], various fractal constructions [8,9,22] or mathematic quasicrystals [10,4].

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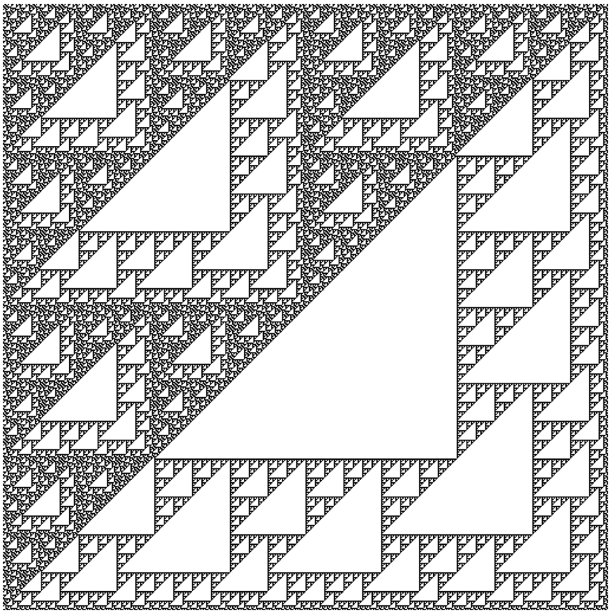

Fig. 1. Thue–Morse–Pascal, 512×512.

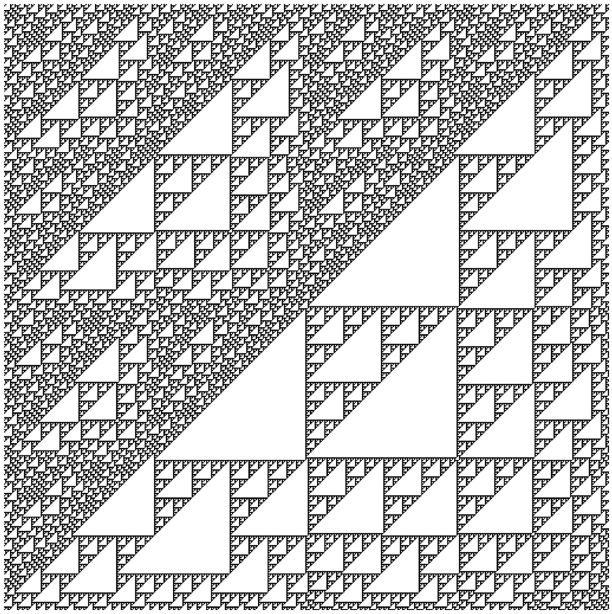

Fig. 2. Doubling-Period-Pascal, 512×512.

Definition 1.1. Let A be a finite set and $f : A^3 \to A$ a fixed function. We call the set A alphabet and the function f recurrence. We will refer to the function f as $f(x, y, z)$. We also fix two sequences $u, v : \mathbb{N} \to A$ with $u(0) = v(0)$, called initial conditions. We say that the tuple (A, f, u, v) defines a recurrent two-dimensional sequence $a : \mathbb{N}^2 \to A$ if the following conditions are fulfilled: (1) $\forall k \in \mathbb{N}$, $a(k, 0) = u(k)$ and $a(0, k) = v(k)$ and (2) $\forall i, j > 0$, $a(i, j) = f(a(i, j-1), a(i-1, j-1), a(i-1, j))$.

In the case that $u = v$ we mention just one of them in the tuple. If u or v are periodic, we just write down the period. We must stress the fact that the notion of recurrence defined here has a very different meaning than those used for one-dimensional sequences and their associated dynamical systems. In that case, recurrence means that each block occurring in the sequence, occurs infinitely often.

The author proved in [15] that recurrent two-dimensional sequences are Turing complete. After understanding the self-similar nature of a narrow class of recurrent two-dimensional sequences in [16], the author finally conjectured that all recurrent two-dimensional sequences given by homomorphisms of finite abelian p-groups and periodic borders (initial conditions) are produced by systems of substitution, see [16–19] and [20].

We observe that every line $a(i, k) \mid i \in \mathbb{N}$ or column $a(k, j) \mid j \in \mathbb{N}$ in a recurrent two-dimensional sequence with periodic borders is ultimately periodic, and could never be an essentially non-periodic sequence – like the Prouhet–Thue–Morse sequence. This means that the cases studied here are definitely not covered by anything studied by the author before.

2. Definitions and main result

Let A be a finite set (alphabet). An n-dimensional sequence is a function $a : \mathbb{N}^n \to A$. An n-dimensional cube is a function $X : [0, d-1]^n \to A$ (also called d^n-cube). We say that X occurs in A at $\vec{u} \in \mathbb{N}^n$ if $a \mid \vec{u} + [0, d-1]^n \equiv X \mid [0, d-1]^n$. For $d \in \mathbb{N}$ we say that $d \mid \vec{u}$ if there is a $\vec{v} \in \mathbb{N}^n$, $\vec{v} = d\vec{u}$. We say that X occurs in a if there is a $\vec{u}$ such that X occurs in a at $\vec{u}$. We say that X occurs at some d-position in a if moreover $d \mid \vec{u}$. Let $s \in \mathbb{N}$ be a natural number $\geqslant 2$ and let $Y : [0, ds-1]^n \to A$ be some $(ds)^n$-cube over A. We recall that the d-block decomposition of Y is the set $B_d(Y)$ of all d^n-cubes occurring in Y in some d-position. We recall that the $2d$-covering of Y is the set $C_d(Y)$ of all $(2d)^n$ cubes occurring in Y in some d-position. We observe that copies of the elements of $C_d(Y)$ cover Y with overlappings. This is a very important difference between $C_d(Y)$ and $B_d(Y)$.

Definition 2.1. Let $d \geqslant 1$ and $s \geqslant 2$ two natural numbers. A system of substitutions of type $d \to sd$ over the finite set A is a tuple of finite sets $(\mathcal{X}, \mathcal{Y}, X_1, \Sigma)$, as follows: $\mathcal{X}$ is a set of d^n-cubes over A. $\mathcal{Y}$ is a set of $(ds)^n$-cubes over A such that for every $Y \in \mathcal{Y}$, $B_d(Y) \subset \mathcal{X}$. $X_1 \in \mathcal{X}$ is a special element called start symbol. $\Sigma : \mathcal{X} \to \mathcal{Y}$ is called the set of substitution rules, or simply the substitution. Σ has a natural extension defined on the set of cubes Z such that $B_d(Z) \subseteq \mathcal{X}$. We remark that if $B_d(Z) \subseteq \mathcal{X}$ then $B_d(\Sigma(Z)) \subseteq \mathcal{X}$, so Σ can be applied again to $\Sigma(Z)$. Last but not least, Σ must fulfill the following condition: $\Sigma(X_1) \mid [0, d-1]^n = X_1$. We say that the substitution Σ is expansive. Also, we can call d primary granulation and s factor of expansion.

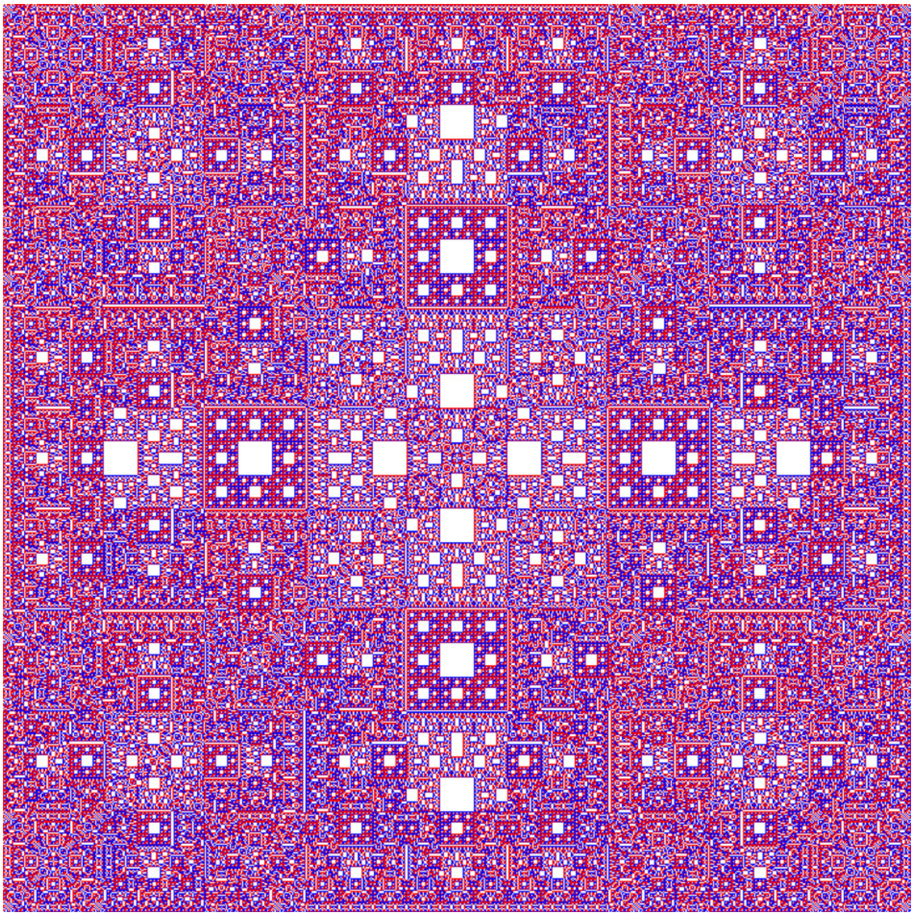

Fig. 3. Essential Heraldry, 729×729.

Definition 2.2. As one immediately can prove by induction, the expansivity of Σ means that for all $m \in \mathbb{N}$ one has that $\Sigma^m(X_1) \mid [0, ds^{m-1}]^n = \Sigma^{m-1}(X_1)$. So we can define the n-dimensional sequence $b := \lim_{m\to\infty} \Sigma^m(X_1)$. We say that the n-dimensional sequence a is defined by substitution.

Theorem 2.3. *Let A be a finite set, $f : A^3 \to A$ a function and $a : \mathbb{N}^2 \to A$ a recurrent two-dimensional sequence satisfying $a(i, j) = f(a(i, j-1), a(i-1, j-1), a(i-1, j))$ for all $i, j \in \mathbb{N}$. Let $(A, \mathcal{X}, \mathcal{Y}, X_1, \Sigma)$ be a substitution of type $d \to sd$ generating a two-dimensional sequence b. If the following conditions are satisfied*: (1) $\forall k \in \mathbb{N}$, $b(k, 0) = a(k, 0)$ *and* $b(0, k) = a(0, k)$, (2) *There exists $M \in \mathbb{N}$ such that $a \mid [0, ds^M - 1] = \Sigma^M(Y_1)$ and $C_d(\Sigma^M(Y_1)) = C_d(\Sigma^{M-1}(Y_1))$; then $a = b$.*

Substitution in multi-dimensional sequences and many aspects of this tool can be also found in [23,24,14,3,1], [2,16,17,19,20] and in the references therein. Theorem 2.3 is more general than the corresponding theorems proven in [17] and [19] because it allows more general borders. This allows new applications, as follows: Let $(t(k))_{k\geqslant 0}$ be the Prouhet–Thue–Morse Sequence, got applying the substitution rules $0 \to 01$ and $1 \to 10$ with start symbol 0 and $(t'(k))_{k\geqslant 0}$ be the complementary Prouhet–Thue–Morse Sequence, got by the same rules but with start symbol 1. Let $(d(k))_{k\geqslant 0}$ the Doubling-Period Sequence, got applying the substitution rules $0 \to 01$ and $1 \to 00$ with start symbol 0 and $(d'(k))_{k\geqslant 0}$ be the complementary Doubling-Period Sequence, got by the rules $1 \to 10$ and $0 \to 11$ with start symbol 1. Then we call the following two-dimensional sequences: $(\mathbb{Z}/2^k\mathbb{Z}, x+z, t, t)$ Thue–Morse–Pascal modulo 2^k, $(\mathbb{Z}/2^k\mathbb{Z}, x+z, t', t')$ complementary Thue–Morse–Pascal modulo 2^k, $(\mathbb{Z}/2^k\mathbb{Z}, x+z, d, d)$ Doubling-Period-Pascal modulo 2^k, $(\mathbb{Z}/2^k\mathbb{Z}, x+z, d', d')$ complementary Doubling-Period-Pascal modulo 2^k, $(\mathbb{Z}/2^k\mathbb{Z}, x+z, t, d)$ TMDP modulo 2^k, and finally $(\mathbb{Z}/3\mathbb{Z}, x+y+z, \{0, 0 \to 010, 1 \to 111\})$ Essential Heraldry. In the case that $k = 1$ we do not say “modulo 2” anymore, and speak only about the Thue–Morse–Pascal two-dimensional sequence, etc.

The following results can be automatically proven using a computer and the principle expressed by Theorem 2.3: (1) Both Thue–Morse–Pascal (Fig. 1) and the complementary Thue–Morse–Pascal are produced by systems of substitution of type $4 \to 8$ with 15 rules. (2) Both Doubling-Period-Pascal (Fig. 2) and the complementary Doubling-Period-Pascal are produced by systems of substitution of type $8 \to 16$ with 70 rules. (3) TMDP is produced by a system of substitution of type $4 \to 8$ with 47 rules. (4) Thue–Morse–Pascal modulo 4 is produced by a system of substitutions of type $8 \to 16$ with 284 rules. (5) TMDP modulo 4 is produced by a system of substitutions of type $8 \to 16$ with 1712 rules. (6) Essential Heraldry (Fig. 3) is produced by a system of substitutions of type $3 \to 9$ with 171 rules.

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