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Holomorphic Cartan geometries on uniruled surfaces

Géométries de Cartan holomorphes des surfaces uniréglées

Benjamin McKay

School of Mathematical Sciences, University College Cork, Cork, Ireland

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ABSTRACT

We classify holomorphic Cartan geometries on every compact complex surface which contains a rational curve.

RÉSUMÉ

Dans cette Note nous classifions les géométries de Cartan holomorphes sur toute surface complexe compacte contenant une courbe rationnelle.

1. Introduction

In my work with Indranil Biswas [1] it became clear that the classification of holomorphic Cartan geometries on uniruled compact complex surfaces was in sight. In this paper, I find that classification. The classification is essential for a paper in progress in which Sorin Dumitrescu and I will survey the holomorphic locally homogeneous structures on all compact complex surfaces. The main result in the paper you have before you is that if a compact complex surface S bears a holomorphic Cartan geometry, and contains a rational curve, then the Cartan geometry is flat, and either S is a rational homogeneous variety with its standard flat holomorphic Cartan geometry, or S is a flat $\mathbb{P}^1$-bundle $S \to C$ and the holomorphic Cartan geometry is induced by a locally homogeneous geometric structure on C. We will see explicit examples of complex manifolds on which the moduli stack of holomorphic Cartan geometries with a fixed model is not a complex analytic space (or orbispace). We also see in explicit examples that deformations of holomorphic Cartan geometries can give rise to nontrivial deformations of the total space of the Cartan geometry as a holomorphic principal bundle.

A more detailed version of this paper appears on the arXiv.

2. Definitions

We will refer to (G, X)-structures [2] where $X = G/H$ is homogeneous as G/H-structures.

2.1. Quotienting by the kernel

Definition 1. (See Sharpe [6].) If $H \subset G$ is a closed subgroup of a Lie group, the *kernel* of the pair (G, H) is

$$K = \bigcap_{g \in G} gHg^{-1},$$

E-mail address: b.mckay@ucc.ie.

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i.e. the largest subgroup of H which is normal in G. The kernel is precisely the set of elements of G which act trivially on G/H.

Example 1. If (G, H) has kernel K, we can let $\bar{G} = G/K$, $\bar{H} = H/K$, make the obvious morphism $g \in G \mapsto \bar{g} = gK \in \bar{G}$, and then clearly $G/H = \bar{G}/\bar{H}$. Every G/H-structure then has induced $\bar{G}/\bar{H}$-structure, with the same charts, called the *induced effective structure*. Any developing map δ and holonomy morphism h for the G/H-structure gives the obvious developing map $\bar{\delta} = \delta$ and holonomy morphism $\bar{h}$: the composition of h with $G \to \bar{G}$.

A Cartan geometry modelled on a homogeneous space G/H will also be called a G/H-geometry.

3. Example

Example 2. Pick any countable abelian group $A \subset \mathbb{C}^{\times}$ with the discrete topology. Set $G = A \ltimes \mathbb{C}$ and $H = A$. The group G acts holomorphically, faithfully and transitively on $X = \mathbb{C} = G/H$. Every complex Lie group G acting holomorphically, faithfully and transitively on $\mathbb{C}$ and not equal to the affine group of $\mathbb{C}$ has this form. The group A need not be finitely generated; for example A could be the group of all roots of 1. Every translation structure on any elliptic curve is a G/H-structure.

Let

$$A' = \left\{\alpha \in \mathbb{C} \;\middle|\; e^{\alpha} \in A\right\},$$

giving the exact sequence of abelian groups $0 \to \mathbb{Z} \to A' \to A \to 0$. Pick any elliptic curve $C = \mathbb{C}/\Lambda$. A *grain* for Λ and A is a complex number $c \neq 0$ so that $c\Lambda \subset A'$. (If $A' \subset \mathbb{C}$ is a lattice, then a grain is a covering map $C \to \mathbb{C}/A'$.) Suppose that c is a grain. Then for any constant $k \neq 0$, the developing map

$$\delta: z \in \mathbb{C} \mapsto ke^{cz} \in \mathbb{C}$$

and holonomy morphism

$$h: \lambda \in \Lambda \mapsto \left(e^{c\lambda}, 0\right) \in G$$

give a G/H-structure on C. Let $\Gamma(\Lambda, A)$ be the set of grains. Every G/H-structure is a translation structure or constructed from a grain. The moduli space of G/H-structures on C is

$$\left(H^0(C, \kappa)/A\right) \cup \bigcup_{c \in \Gamma(\Lambda, A)} \mathbb{C}^{\times}/A.$$

The automorphisms of a G/H-structure induced from a translation structure are the automorphisms of $\mathbb{C}/\Lambda$, say $z \mapsto az + b$, for which $a \in A$ and $a\Lambda = \Lambda$. So the automorphism group is $\mathbb{Z}_n \ltimes \mathbb{C}$ for some integer $n = 1, 2, 3, 4$ or 6.

The automorphisms of a G/H-structure with grain $c \neq 0$ are the translations $z \in \mathbb{C}/\Lambda$ for which $e^{cz} \in A$, a finite group. In particular, these G/H-structures are not homogeneous.

4. Lifting from curves to surfaces

Definition 2. Suppose that $E \to M'$ is a holomorphic Cartan geometry modelled on a complex homogeneous space G/H', with Cartan connection ω. Suppose that $H \subset H'$ is a closed complex Lie subgroup. Let $M = E/H$. Then $E \to M$ is a holomorphic Cartan geometry modelled on G/H, with Cartan connection ω. We say that the Cartan geometry on M is the *lift* of the Cartan geometry on M'.

Definition 3. Suppose that G is a complex Lie group and that $H \subset H' \subset G$ are closed complex subgroups. Suppose that $\dim G/H = 2$ and $\dim G/H' = 1$. Suppose that C is a complex curve. Let $\pi = \pi_1(C)$. Suppose that we have a G/H'-structure on C, with developing map $\delta_C : \tilde{C} \to G/H'$, and holonomy morphism $h_C : \pi \to G$. Take the bundle $G/H \to G/H'$ and let

$$\tilde{S} = \delta_C^* G/H,$$

so that we have the diagram

$$\begin{array}{ccc} \tilde{S} & \longrightarrow & G/H \\ \downarrow & & \downarrow \\ \tilde{C} & \longrightarrow & G/H' \\ \downarrow & & \\ C & & \end{array}$$

So $\tilde{S}$ is a π-invariant surface inside $\tilde{C} \times G/H$. We then let $S = \pi \backslash \tilde{S}$, so that S is a complex surface, and $S \to C$ is a holomorphic fiber bundle, with fibers isomorphic to H'/H. The surface $\tilde{S}$ is a covering space of S, and S has a G/H-structure with developing map $\delta_S : \tilde{S} \to G/H$ as above, and holonomy morphism $h_S : \pi_1(S) \to G$ given by the composition

$$\pi_1(S) \to \pi \to G.$$

We will say that this G/H-structure on S is the *lift* of the G/H'-structure on C. (This is a special case of a lift of a Cartan geometry.)

Example 3. We define a lift of any structure on a curve to a structure on any flat $\mathbb{P}^1$-bundle over that curve. Suppose that C is a compact complex curve, and that G_0/H_0 is a complex-homogeneous curve. Let $\pi = \pi_1(C)$. Take a G_0/H_0-structure on C, say with developing map

$$\delta_C : \tilde{C} \to G_0/H_0$$

and holonomy morphism

$$h_C : \pi \to G_0.$$

Take any group morphism

$$\rho : \pi \to \mathbb{P}\mathrm{SL}(2, \mathbb{C}).$$

Let π act on $\tilde{C} \times \mathbb{P}^1$ by

$$\gamma(z, w) = \big(\gamma z, \rho(\gamma) w\big).$$

Define a compact complex surface S by

$$S = \tilde{C} \times_\pi \mathbb{P}^1.$$

Let $B \subset \mathbb{P}\mathrm{SL}(2, \mathbb{C})$ be the stabilizer of a point of $\mathbb{P}^1$. Let $G = G_0 \times \mathbb{P}\mathrm{SL}(2, \mathbb{C})$, and $H = H_0 \times B$. On S, define a G/H-structure, by taking as developing map

$$\delta_S : \tilde{S} = \tilde{C} \times \mathbb{P}^1 \to G/H = (G_0/H_0) \times \mathbb{P}^1,$$

the map

$$\delta_S(z, w) = \big(\delta_C(z), w\big),$$

and as holonomy morphism

$$h_S : \pi_1(S) = \pi \to G = G_0 \times \mathbb{P}\mathrm{SL}(2, \mathbb{C})$$

the map

$$h_S(\gamma) = \big(h_C(\gamma), \rho(\gamma)\big).$$

Theorem 1. *Suppose that G/H is a connected complex-homogeneous surface. Suppose that S is a compact complex surface containing a rational curve and that S has an effective holomorphic Cartan geometry modelled on G/H. Then the Cartan geometry is flat, a G/H-structure. Up to isomorphism, either*

(1) $S = \mathbb{P}^2$ *with the standard projective structure or*
(2) $S = \mathbb{P}^1 \times \mathbb{P}^1$ *with the standard G/H-structure where*

$$G = \mathbb{Z}_2 \ltimes \big(\mathbb{P}\mathrm{SL}(2, \mathbb{C}) \times \mathbb{P}\mathrm{SL}(2, \mathbb{C})\big),$$

or

(3) *the G/H-structure on S, after perhaps quotienting out the kernel of G/H, is constructed as in Example* 3. *The moduli space is then the product of the moduli space of G_0/H_0-structures on C with the representation variety*

$$\mathrm{Hom}\big(\pi_1(C), \mathbb{P}\mathrm{SL}(2, \mathbb{C})\big)/\mathbb{P}\mathrm{SL}(2, \mathbb{C}).$$

Proof. The deformation space of any rational curve in any compact complex surface has compact components. By the main theorem of Biswas and McKay [1], since our surface S contains a rational curve, there is some closed complex subgroup $H' \subset G$ and a compact complex manifold C with $\dim C < \dim S$, and a fiber bundle morphism $S \to C$ with rational homogeneous fibers, and a holomorphic Cartan geometry on C modelled on G/H', so that the holomorphic Cartan geometry on S is lifted from that on C. Moreover, H'/H must be a connected rational homogeneous variety. Every holomorphic Cartan geometry on a complex curve is flat, so is lifted from a unique G/H'-structure with developing map and holonomy morphism.

If C of dimension 0, then C is a point, $H' = G$, $S = G/H$ and the holomorphic Cartan geometry on S is the standard G/H-structure on G/H. But then also we must have H'/H a rational homogeneous variety, so $S = \mathbb{P}^1 \times \mathbb{P}^1$ or $S = \mathbb{P}^2$ with its standard G/H-structure. If $S = \mathbb{P}^2$, then the group G, to act faithfully and transitively, must be $\mathbb{P}\,\mathrm{SL}(3,\mathbb{C})$ and H must be the Borel subgroup of $\mathbb{P}\,\mathrm{SL}(3,\mathbb{C})$. For $S = \mathbb{P}^1 \times \mathbb{P}^1$, in order that G act transitively on G/H, we can have either

$$G = \mathbb{Z}_2 \ltimes \big(\mathbb{P}\,\mathrm{SL}(2,\mathbb{C}) \times \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})\big)$$

or the subgroup

$$G = \mathbb{P}\,\mathrm{SL}(2,\mathbb{C}) \times \mathbb{P}\,\mathrm{SL}(2,\mathbb{C}).$$

Henceforth we can assume that C is of dimension 1, so $H'/H = \mathbb{P}^1$ and G/H is a complex-homogeneous surface, invariantly ruled. By the classification of complex-homogeneous surfaces [3], if a complex-homogeneous surface is ruled, then it is a product $G/H = (G_0/H_0) \times \mathbb{P}^1$ where G_0/H_0 is a complex-homogeneous complex curve, acted on transitively by G. It is easy to check from the classification of complex-homogeneous surfaces [4,5] that G must be a product

$$G = G_0 \times \mathbb{P}\,\mathrm{SL}(2,\mathbb{C}),$$

and so

$$H = H_0 \times B,$$

where B is the Borel subgroup in $\mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$. The group H' must contain H and have $H'/H = \mathbb{P}^1$, so H' must contain an image of $\mathrm{SL}(2,\mathbb{C})$. Since $\mathrm{SL}(2,\mathbb{C})$ has no nontrivial morphism to $\mathrm{Bihol}(C)$, we have $\{1\} \times \mathbb{P}\,\mathrm{SL}(2,\mathbb{C}) \subset H'$, and therefore $H' = H_0 \times \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$.

The universal covering space of S is

$$\begin{aligned}\tilde{S} &= \delta_C^*(G/H)\\ &= \delta_C^*\big((G_0/H_0) \times \mathbb{P}^1\big)\\ &= \big(\delta_C^*(G_0/H_0)\big) \times \mathbb{P}^1\\ &= \tilde{C} \times \mathbb{P}^1.\end{aligned}$$

The developing map δ_S is then by definition the identity map on $* \times \mathbb{P}^1$. □

5. Conclusion

We have found that for compact complex surfaces which contain rational curves, essentially there are no holomorphic Cartan geometries, except for a trivial construction from holomorphic locally homogeneous geometric structures on curves. On compact complex surfaces without rational curves, and in higher dimensions, more complicated phenomena show up.

In our examples above, we see that the moduli stack of holomorphic Cartan geometries on a fixed complex manifold, with a fixed model G/H, might not be a complex analytic space or orbispace. We can also see that deformations of a holomorphic Cartan geometry can give rise to nontrivial deformations of the total space of the Cartan geometry as a holomorphic principal bundle: take the affine structure of an elliptic curve, and a family of representations ρ_t of its fundamental group into $\mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$, so that ρ_{t_1} and ρ_{t_2} are not conjugate for generic values t_1 and t_2. Build a holomorphic Cartan geometry on each of the associated flat $\mathbb{P}^1$-bundles.

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