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Numerical solutions of backward stochastic differential equations: A finite transposition method

Solutions numériques des équations différentielles stochastiques rétrogrades : « A finite transposition method »

Penghui Wang [a], Xu Zhang [b,c]

[a] *School of Mathematics, Shandong University, Jinan 250100, China*
[b] *Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*
[c] *Yangtze Center of Mathematics, Sichuan University, Chengdu 610064, China*

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ABSTRACT

In this Note, we present a new numerical method for solving backward stochastic differential equations. Our method can be viewed as an analogue of the classical finite element method solving deterministic partial differential equations.

RÉSUMÉ

Dans cette Note, nous présentons une nouvelle méthode pour résoudre numériquement les équations différentielles stochastiques rétrogrades. Notre méthode ressemble à la méthode des éléments finis qui permet de résoudre numériquement les équations aux dérivées partielles déterministes.

1. Introduction

Linear and nonlinear Backward Stochastic Differential Equations (BSDEs in short) were introduced in [1] and [9], respectively. It is well-known that BSDE plays crucial roles in Stochastic Control, Mathematical Finance etc. Clearly, for applications, it deserves to develop effective numerical methods for BSDEs.

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$, on which a 1-dimensional standard Brownian motion $\{w(t)\}_{t\in[0,T]}$ is defined. We denote by $L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ $(n\in\mathbb{N})$ the Hilbert space consisting of all $\mathcal{F}_t$-measurable ($\mathbb{R}^n$-valued) square integrable random variables; by $L^2_{\mathbb{F}}(\Omega; L^r(0,T;\mathbb{R}^n))$ $(1 \leqslant r \leqslant \infty)$ the Banach space consisting of all $\mathbb{R}^n$-valued $\{\mathcal{F}_t\}$-adapted processes $X(\cdot)$ such that $\mathbb{E}|X(\cdot)|^2_{L^r(0,T;\mathbb{R}^n)} < \infty$; and by $L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n))$ the Banach space consisting of all $\mathbb{R}^n$-valued $\{\mathcal{F}_t\}$-adapted càdlàg processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^n)}) < \infty$. For $y_T \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ and $f(\cdot,\cdot,\cdot)$ satisfies $f(\cdot,0,0) \in L^2(\Omega; L^1(0,T;\mathbb{R}^n))$ and the usual globally Lipschitz condition, we consider the following BSDE

$$\begin{cases} \mathrm{d}y(t) = f(t, y(t), Y(t))\,\mathrm{d}t + Y(t)\,\mathrm{d}w(t) \quad \text{in } [0,T], \\ y(T) = y_T. \end{cases} \tag{1}$$

E-mail addresses: phwang@sdu.edu.cn (P. Wang), xuzhang@amss.ac.cn (X. Zhang).

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Various numerical methods have been developed to solve Eq. (1), say in [2,7,10,12] and the references therein. These methods use essentially the strong form of (1), which holds true only if $\mathbb{F}$ is the natural filtration generated by the Brownian motion. Also, it seems that the previous methods need to compute the conditional expectation, which is in general not easy to be furnished numerically.

In this Note, we shall present a new numerical method solving BSDEs from the viewpoint of transposition solution introduced in [6], as recalled below.

Definition 1.1. A couple $(y(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ is called a transposition solution of BSDE (1), if for any $t \in [0,T]$, $(u(\cdot), v(\cdot), \eta) \in L^2_{\mathbb{F}}(\Omega; L^1(t,T;\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(t,T;\mathbb{R}^n)) \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, the following variational equation holds

$$\mathbb{E}\langle z(T), y_T\rangle - \mathbb{E}\langle \eta, y(t)\rangle = \mathbb{E}\int_t^T \big\langle z(\tau), f\big(\tau, y(\tau), Y(\tau)\big)\big\rangle \,\mathrm{d}\tau + \mathbb{E}\int_t^T \big\langle u(\tau), y(\tau)\big\rangle \,\mathrm{d}\tau + \mathbb{E}\int_t^T \big\langle v(\tau), Y(\tau)\big\rangle \,\mathrm{d}\tau, \tag{2}$$

where $z(\tau) = \eta + \int_t^\tau u(s)\,\mathrm{d}s + \int_t^\tau v(s)\,\mathrm{d}w(s)$.

We refer to [6] for the well-posedness of Eq. (1) in the sense of transposition solution. It is easy to see that, if this equation admits a strong solution (say under the assumption of natural filtration), it coincides with the transposition solution.

Our numerical schemes for solving Eq. (1) can be described as follows.

1) Take a suitable finite-dimensional subspace H_m of $L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$, where m is a natural number;
2) If the solution of (1) exists, then it should satisfy the variational equation (2) for any $u, v \in H_m$. By taking u and v to be the orthonormal basis of H_m, we obtain a system of approximating equations;
3) If the solution of the system of approximating equations exists, then we find a class of numerical solutions of (1);
4) Finally, we show the convergence of the above numerical solutions.

Clearly, the above procedure is, in spirit, very close to that of the classical finite element method solving deterministic PDEs (e.g., [3]). Therefore, our method to solve BSDEs can be viewed as a stochastic version of the finite element-type method. Nevertheless, the notion of "stochastic finite element method" has already been used for other purpose, say [4,5,8] and references therein for solving random PDEs. Note also that our method is quite different from that in these references, and therefore instead we call it a finite transposition method.

There are at least two reasons for us to develop this new numerical approach for BSDE. The first one is that, we can solve the BSDE with general filtration. The second is that, in our approach, we do not need to compute the conditional expectation.

We refer to [11] for the details of the proofs of the results in this Note and other results in this context.

2. Numerical schemes and convergence

For simplicity, we consider only the following linear BSDE (with $f(\cdot) \in L^2(\Omega; L^1(0,T;\mathbb{R}^n))$)

$$\begin{cases} \mathrm{d}y(t) = f(t)\,\mathrm{d}t + Y(t)\,\mathrm{d}w(t), & t \in [0,T), \\ y(T) = y_T. \end{cases} \tag{3}$$

Assume that $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ is a separable Hilbert space. For any $N \in \mathbb{N}$, write $\mathfrak{R}_N = \{t_\ell \mid t_\ell = \frac{\ell}{2^N}T,\ \ell = 0,\dots,2^N\}$. For any $k \in \{0,\dots,2^N-1\}$, define a sequence of simple processes $\{e_{ki}(\cdot,\cdot)\}_{i=1}^{M_{k,N}}$ by

$$e_{ki}(t,\omega) = \begin{cases} \chi_{[t_k,t_{k+1})}(t) h_{ki}(\omega), & 0 \leqslant k < 2^N - 1, \\ \chi_{[t_k,T]}(t) h_{ki}(\omega), & k = 2^N - 1, \end{cases} \tag{4}$$

where $\{M_{0,N}, M_{1,N}, \dots, M_{2^N-1,N}\}$ is an increasing sequence of integers. Since $L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n) \subseteq L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n)$ (for any $0 \leqslant t < s \leqslant T$), we may assume that $\{h_{ki}\}$ satisfy the following:

1) For any fixed $k \in \{0,\dots,2^N-1\}$, $\{h_{ki}\}_{i=0}^{M_{k,N}}$ is an orthogonal set in $L^2_{\mathcal{F}_{t_k}}(\Omega;\mathbb{R}^n)$, with the norm $|h_{ki}|_{L^2_{\mathcal{F}_{t_k}}(\Omega;\mathbb{R}^n)} = \sqrt{\frac{2^N}{T}}$, and hence $|e_{ki}|_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))} = 1$;
2) If $0 \leqslant k < l \leqslant 2^N - 1$, then $\{h_{ki}\}_{i=0}^{M_{k,N}} \subset \{h_{li}\}_{i=0}^{M_{l,N}}$; and
3) If $s_0 = \frac{k_0}{2^{N_0}}T$ for some $N_0 \in \mathbb{N}$ and $k_0 \in \{0,\dots,2^{N_0}-1\}$, then $s_0 = 2^{l-N_0}k_0T/2^l \in \mathfrak{R}_l$ for any $l \geqslant N_0$. For $l \geqslant N_0$, write $k_l = 2^{l-N_0}k_0$. Then, $\{h_{k_j i}\}_{i=0}^{M_{k_j,j}} \subset \{h_{k_l i}\}_{i=0}^{M_{k_l,l}}$ for $N_0 \leqslant j < l$, and $\bigcup_{l=N_0}^\infty \{h_{k_l i}\}_{i=1}^{M_{k_l,l}}$ is an orthogonal basis of $L^2_{\mathcal{F}_{s_0}}(\Omega;\mathbb{R}^n)$.

Denote by H_N the subspace of $L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ spanned by $\{e_{0i}\}_{i=1}^{M_{0,N}}, \ldots, \{e_{2^N-1,i}\}_{i=1}^{M_{2^N-1,N}}$. This is the finite element subspace that we will employ below. Replace $L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ by H_N, then the desired numerical scheme follows by trying to find $y_N, Y_N \in H_N$ such that (2) (with $f(\cdot, y(\cdot, Y(\cdot))$ replacing by $f(\cdot)$) holds for $\eta = 0$ and for all $u, v \in H_N$.

To find the y_N in H_N, suppose $y_N = \sum_{k=0}^{2^N-1}\sum_{i=0}^{M_{k,N}} \alpha_{ki} e_{ki}$. Choosing $u = e_{ki}$, $v = 0$ and $\eta = 0$, we get $z_{ki}(t) = \int_0^t e_{ki}(\tau)\,\mathrm{d}\tau$, and hence $\mathbb{E}\langle z_{ki}(T), y_T\rangle = \mathbb{E}\int_0^T \langle z_{ki}(\tau), f(\tau)\rangle\,\mathrm{d}\tau + \sum_{l,j} \alpha_{lj}\mathbb{E}\int_0^T \langle e_{ki}(\tau), e_{lj}(\tau)\rangle\,\mathrm{d}\tau$. Since $\{e_{ki}\}$ is an orthonormal basis of H_N, it follows that $\mathbb{E}\int_0^T \langle e_{lj}(\tau), e_{ki}(\tau)\rangle\,\mathrm{d}\tau = \delta_{kl}\delta_{ij}$. Therefore,

$$\alpha_{ki} = \frac{T}{N}\mathbb{E}\langle h_{ki}, y_T\rangle - \mathbb{E}\int_0^T \big\langle (\tau \wedge t_{k+1} - \tau \wedge t_k) h_{ki}, f(\tau)\big\rangle\,\mathrm{d}\tau. \tag{5}$$

Similarly, suppose $Y_N = \sum_{k=0}^{2^N-1}\sum_{i=0}^{M_{k,N}} \beta_{ki} e_{ki}$. By taking $u = 0$, $\eta = 0$ and $v = e_{ki}$ to get a corresponding $z_{ki}(t) = \int_0^t e_{ki}(\tau)\,\mathrm{d}w\tau$, we find that

$$\beta_{ki} = \mathbb{E}\big\langle \big(w(t_{k+1}) - w(t_k)\big) h_{ki}, y_T\big\rangle - \mathbb{E}\int_0^T \big\langle \big(w(\tau \wedge t_{k+1}) - w(\tau \wedge t_k)\big) h_{ki}, f(\tau)\big\rangle\,\mathrm{d}\tau.$$

We now show the convergence of the sequence $\{(y_N, Y_N)\}$ of numerical solutions constructed above.

Theorem 2.1. *Let (y, Y) be the transposition solution of (3). Then y_N and Y_N are projections of y (viewing as an element of $L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$) and Y onto H_N, respectively.*

By the construction of H_N, it is clear that $H_N \subset H_M$ provided $N < M$. Moreover, $\bigcup_{N=1}^{\infty} H_N$ is dense in $L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$. Hence,

$$\mathbb{E}\int_0^T \big|y_N(\tau) - y(\tau)\big|^2\,\mathrm{d}\tau + \mathbb{E}\int_0^T \big|Y_N(\tau) - Y(\tau)\big|^2\,\mathrm{d}\tau \to 0, \quad \text{as } N \to \infty. \tag{6}$$

Furthermore, starting from (6), we can show the following convergent result.

Theorem 2.2. *As $N \to \infty$, (y_N, Y_N) tends to (y, Y) in $L^2_{\mathbb{F}}(\Omega; D([0,T],\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$.*

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