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Differential Geometry

On m-th root metrics with special curvature properties

Sur les métriques racines m-ièmes ayant des propriétés de courbure spéciales

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ABSTRACT

In this Note, we prove that every m-th root Finsler metric with isotropic Landsberg curvature reduces to a Landsberg metric. Then, we show that every m-th root metric with almost vanishing **H**-curvature has vanishing **H**-curvature.

RÉSUMÉ

Dans cette Note, nous montrons que toutes les métriques de Finsler racines m-ièmes ayant une courbure de Landsberg isotrope se réduisent à une métrique de Landsberg. Nous montrons ensuite que toutes les métriques de Finsler racines m-ièmes ayant une **H**-courbure presque nulle ont en fait une **H**-courbure nulle.

1. Introduction

The m-th root Finsler metrics originated from Riemann's celebrated address "On the hypothesis, which lie the foundation of geometry", made in 1854. This class of metrics is regarded as a direct generalization of the class of Riemannian metrics in the sense that the second root metric is a Riemannian metric. The third and fourth root metrics are called the cubic metric and quartic metric, respectively [2–4].

Recent research has shown that such metrics have important applications in Biology, Ecology, Physics and information theory. In two papers [5,6], V. Balan and N. Brinzei study the Einstein equations for some relativistic models relying on such metrics. Y. Yu and Y. You show that an m-th root Einstein Finsler metric is Ricci-flat [9]. The authors characterize locally dually flat m-th root Finsler metrics as well as m-th root y-Berwald metrics in [8].

In this Note, we prove that every isotropic Landsberg m-th root metric is a Landsberg metric. Then, we show that every m-th root Finsler metric with almost vanishing **H**-curvature has vanishing **H**-curvature.

Let M be an n-dimensional C^∞ manifold. Denote by $TM = \bigcup_{x\in M} T_xM$ the tangent space of M. Let $TM_0 = TM \setminus \{0\}$. Let $F = \sqrt[m]{A}$ be a Finsler metric on M, where A is given by

$$A := a_{i_1\dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m} \tag{1}$$

with $a_{i_1\dots i_m}$ symmetric in all its indices [8]. Then F is called an m-th root Finsler metric. Let F be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Put

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$$A_i=\frac{\partial A}{\partial y^i},\qquad A_{ij}=\frac{\partial^2 A}{\partial y^i\partial y^j},\qquad A_{x^k}=\frac{\partial A}{\partial x^k},\qquad A_0=A_{x^k}y^k,\qquad A_{0j}=A_{x^k y^j}y^k.$$

Suppose that (A_{ij}) is a positive definite tensor and (A^{ij}) denotes its inverse. Then the following hold:

$$g_{ij}=\frac{A^{\frac{2}{m}-2}}{m^2}\big[mAA_{ij}+(2-m)A_iA_j\big],\qquad g^{ij}=A^{-\frac{2}{m}}\bigg[mAA^{ij}+\frac{m-2}{m-1}y^iy^j\bigg],\tag{2}$$

$$y^iA_i=mA,\qquad y^iA_{ij}=(m-1)A_j,\qquad y_i=\frac{1}{m}A^{\frac{2}{m}-1}A_i,\qquad A^{ij}A_i=\frac{1}{m-1}y^j,\qquad A_iA_jA^{ij}=\frac{m}{m-1}A.\tag{3}$$

Let (M,F) be a Finsler manifold. The second derivatives of $\frac{1}{2}F_x^2$ at $y\in T_xM_0$ are the components of an inner product $\mathbf{g}_y$ on T_xM. The third order derivatives of $\frac{1}{2}F_x^2$ at $y\in T_xM_0$ are a symmetric trilinear form $\mathbf{C}_y$ on T_xM. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively. The rate of change of the Cartan torsion along geodesics is the *Landsberg curvature* $\mathbf{L}_y$ on T_xM for any $y\in T_xM_0$. F is said to be Landsbergian if $\mathbf{L}=0$. The quotient $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of Cartan torsion $\mathbf{C}$ along Finslerian geodesics. Then F is said to be isotropic Landsberg metric if $\mathbf{L}=cF\mathbf{C}$, where $c=c(x)$ is a scalar function on M. In this paper, we prove the following:

Theorem 1. *Let (M,F) be an n-dimensional m-th root Finsler manifold. Suppose that F is a non-Riemannian isotropic Landsberg metric. Then F reduces to a Landsberg metric.*

Let F be a Finsler metric on a manifold M. The geodesics of F are characterized locally by the equations $\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}+2G^i(x,\frac{\mathrm{d}x}{\mathrm{d}t})=0$, where $G^i=\frac{1}{4}g^{ik}\{2\frac{\partial g_{pk}}{\partial x^q}-\frac{\partial g_{pq}}{\partial x^k}\}y^py^q$ are coefficients of the spray associated with F. A Finsler metric F is called a Berwald metric if $G^i=\frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ are quadratic in $y\in T_xM$ for any $x\in M$. Taking the trace of Berwald curvature gives rise to the mean Berwald curvature $\mathbf{E}$. In [1], Akbar-Zadeh introduces the non-Riemannian quantity $\mathbf{H}$ which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. More precisely, the non-Riemannian quantity $\mathbf{H}=H_{ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j$ is defined by $H_{ij}:=E_{ij|s}y^s$. He proves that for a Finsler manifold of scalar flag curvature $\mathbf{K}$ with dimension $n\geqslant 3$, $\mathbf{K}=constant$ if and only if $\mathbf{H}=0$. It is remarkable that the Riemann curvature $R_y=R^i{}_k\,\mathrm{d}x^k\otimes\frac{\partial}{\partial x^i}|_x:T_xM\to T_xM$ is a family of linear maps on tangent spaces, defined by

$$R^i{}_k=2\frac{\partial G^i}{\partial x^k}-y^j\frac{\partial^2G^i}{\partial x^j\partial y^k}+2G^j\frac{\partial^2G^i}{\partial y^j\partial y^k}-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

A Finsler metric F is said to be of scalar curvature if there is a scalar function $\mathbf{K}=\mathbf{K}(x,y)$ such that $R^i{}_k=\mathbf{K}(x,y)F^2h^i_k$. If $\mathbf{K}=constant$, then F is called of constant flag curvature.

A Finsler metric is called of almost vanishing $\mathbf{H}$-curvature if $H_{ij}=\frac{n+1}{2F}\theta h_{ij}$, for some 1-form θ on M, where h_{ij} is the angular metric. It is remarkable that in [7], Z. Shen with the authors prove that every Finsler metric of scalar flag curvature $\mathbf{K}$ and of almost vanishing $\mathbf{H}$-curvature has almost isotropic flag curvature, i.e., the flag curvature is in the form $\mathbf{K}=\frac{3\theta}{F}+\sigma$, for some scalar function σ on M.

Theorem 2. *Let (M,F) be an n-dimensional m-th root manifold with $n\geqslant 2$. Suppose that F has almost vanishing $\mathbf{H}$-curvature. Then $\mathbf{H}=0$.*

2. Proof of the Main Theorems

Lemma 3. *(See [9].) Let F be an m-th root Finsler metric on an open subset $U\subset\mathbb{R}^n$. Then the spray coefficients of F are given by*

$$G^i=\frac{1}{2}(A_{0j}-A_{x^j})A^{ij}.\tag{4}$$

Proof of Theorem 1. Let $F=\sqrt[m]{A}$ be an m-th root isotropic Landsberg metric, i.e., $L_{ijk}=cFC_{ijk}$, where $c=c(x)$ is a scalar function on M. The Cartan tensor of F is given by the following:

$$C_{ijk}=\frac{1}{m}A^{\frac{2}{m}-3}\bigg[A^2A_{ijk}+\Big(\frac{2}{m}-1\Big)\Big(\frac{2}{m}-2\Big)A_iA_jA_k+\Big(\frac{2}{m}-1\Big)A\{A_iA_{jk}+A_jA_{ki}+A_kA_{ij}\}\bigg].\tag{5}$$

Since $L_{ijk}=-\frac{1}{2}y_sG^s{}_{y^iy^jy^k}$, then we have $L_{ijk}=-\frac{1}{2m}A^{\frac{2}{m}-1}A_sG^s{}_{y^iy^jy^k}$. Therefore, we get

$$A_sG^s{}_{y^iy^jy^k}=-2cA^{\frac{1}{m}-2}\bigg[A^2A_{ijk}+\Big(\frac{2}{m}-1\Big)\bigg\{\Big(\frac{2}{m}-2\Big)A_iA_jA_k+A\{A_iA_{jk}+A_jA_{ki}+A_kA_{ij}\}\bigg\}\bigg].\tag{6}$$

By Lemma 3, the left-hand side of (6) is a rational function in y, while its right-hand side is an irrational function in y. Thus, either $c = 0$ or A satisfies the following PDEs

$$A^2 A_{ijk} + \left(\frac{2}{m} - 1\right)\left(\frac{2}{m} - 2\right) A_i A_j A_k + \left(\frac{2}{m} - 1\right) A\{A_i A_{jk} + A_j A_{ki} + A_k A_{ij}\} = 0. \tag{7}$$

Plugging (7) into (5) implies that $C_{ijk} = 0$. Hence, by Deicke's theorem, F is Riemannian metric, which contradicts our assumption. Therefore, $c = 0$. This completes the proof. □

Proof of Theorem 2. Let $F = \sqrt[m]{A}$ be of almost vanishing **H**-curvature, i.e.,

$$H_{ij} = \frac{n+1}{2F} \theta h_{ij}, \tag{8}$$

where θ is a 1-form on M. The angular metric $h_{ij} = g_{ij} - F^2 y_i y_j$ is given by the following

$$h_{ij} = \frac{A^{\frac{2}{m}-2}}{m^2} \left[m A A_{ij} + (1-m) A_i A_j \right]. \tag{9}$$

Plugging (9) into (8), we get

$$H_{ij} = \frac{(n+1) A^{\frac{1}{m}-2}}{2m^2} \theta \left[m A A_{ij} + (1-m) A_i A_j \right]. \tag{10}$$

By (4), one can see that H_{ij} is rational with respect to y. Thus, (10) implies that $\theta = 0$ or

$$m A A_{ij} + (1-m) A_i A_j = 0. \tag{11}$$

By (9) and (11), we conclude that $h_{ij} = 0$, which is impossible. Hence $\theta = 0$ and $H_{ij} = 0$. □

By the Schur Lemma, Theorem 2 and Theorem 1.1 of [7], we have the following:

Corollary 4. *Let (M, F^n) be an n-dimensional m-th root Finsler manifold of scalar flag curvature* $\mathbf{K}$ *with* $n \geqslant 3$. *Suppose that the flag curvature is given by* $\mathbf{K} = \frac{3\theta}{F} + \sigma$, *where* θ *is a* 1*-form and* $\sigma = \sigma(x)$ *is a scalar function on* M. *Then* $\mathbf{K} = 0$.

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