



Complex Analysis/Harmonic Analysis

## Extensions of the disc algebra and of Mergelyan's theorem

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## ARTICLE INFO

## Article history:

Received 28 April 2011

Accepted after revision 1 June 2011

Available online 17 June 2011

Presented by Jean-Pierre Kahane

## ABSTRACT

We investigate the uniform limits of the set of polynomials on the closed unit disc  $\bar{D}$  with respect to the chordal metric  $\chi$ . More generally, we examine analogous questions replacing  $\mathbb{C} \cup \{\infty\}$  by other metrizable compactifications of  $\mathbb{C}$ .

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## R É S U M É

A toute compactification métrisable  $S$  du plan complexe, nous associons une extension  $A(D, S)$  de l'algèbre du disque  $A(D)$ . Un cas fondamental est celui où  $S = \mathbb{C} \cup \{\infty\}$ . Nous déterminons l'ensemble de limites uniformes des polynômes sur le disque unité fermé  $\bar{D}$ , par rapport à la métrique chordale; ensuite nous étendons cette étude au cas général.

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## Version française abrégée

Il est connu qu'il existe beaucoup de compactifications métrisables du plan complexe  $\mathbb{C}$ ; par exemple  $\mathbb{C} \cup \{\infty\}$ , le disque de Poincaré, le tore  $T^2$  et toute surface compacte. Soit  $S = \mathbb{C} \cup \mathbb{C}^\infty$  une compactification métrisable de  $\mathbb{C}$  munie d'une métrique  $\rho$ .  $\mathbb{C}$  est un ouvert dense dans  $S$ . Nous identifions les limites uniformes sur le disque unité fermé  $\bar{D}$  des polynômes par rapport à la métrique  $\rho$ . Leur ensemble  $A(D, S)$  contient des fonctions de deux types. Le type fini sont les fonctions holomorphes  $f : D \rightarrow \mathbb{C}$  sur le disque unité ouvert  $D$ , telles que  $\lim_{z \rightarrow \zeta, z \in D} f(z)$  existe dans  $S$  pour tout  $\zeta \in \partial D$ . Il est facile de voir que toutes ces fonctions sont en réalité limites uniformes des polynômes. Evidemment le type fini contient l'algèbre classique  $A(D)$  du disque.

Le type infini contient des fonctions  $f : \bar{D} \rightarrow \mathbb{C}^\infty$  continues sur  $\bar{D}$ ; mais il n'est pas en général vrai que toute telle fonction est une limite de polynômes. Dans chaque cas particulier de compactification  $S$  on doit caractériser les fonctions du type infini appartenant à  $A(D, S)$ . La class  $A(D, S)$  admet une métrique complète naturelle.  $A(D)$  avec sa topologie usuelle est une partie ouverte et dense dans  $A(D, S)$ .

**Proposition.** Soit  $S = \mathbb{C} \cup \mathbb{C}^\infty$  et  $S' = \mathbb{C} \cup \mathbb{C}'^\infty$  deux compactifications métrisables du plan  $\mathbb{C}$ . Alors les énoncés suivants sont équivalents.

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- (1) Il existe un homéomorphisme  $T : S \rightarrow S'$  tel que  $T(w) = w$  pour tout  $w \in \mathbb{C}$ .  
 (2) Il existe un homéomorphisme  $F : A(D, S) \rightarrow A(D, S')$  tel que  $F(P) = P$  pour tout polynôme  $P$ .

**Question.** Soit  $L \subset \mathbb{C}$  un ensemble compact avec  $L^c$  connexe et  $f : L \rightarrow S$  une fonction continue telle que  $f(L^0) \subseteq \mathbb{C}$  et  $f|_{L^0}$  est holomorphe. Est il vrai qu'il existe une suite de polynômes  $P_n$  telle que  $P_n \rightarrow f$  uniformément sur  $L$  par rapport à la métrique  $\rho$  de  $S$  ?

La question précédente admet une réponse positive dans le cas particulier  $L = \overline{\Omega}$ , où  $\Omega$  est l'intérieur d'une courbe de Jordan. La raison est que la fonction de Riemann  $\varphi : D \rightarrow \Omega$  s'étend à un homéomorphisme  $\varphi : \overline{D} \rightarrow \overline{\Omega}$  (Carathéodory [5]). Alors on se ramène au cas  $L = \overline{D}$ , où la réponse est connue et positive.

## 1. Spherical approximation

A classical approximation theorem in Complex Analysis is Mergelyan's theorem [12]: If  $K \subset \mathbb{C}$  is a compact set with  $K^c$  connected and if  $f : K \rightarrow \mathbb{C}$  is continuous on  $K$  and holomorphic in  $K^0$  (i.e.  $f \in A(K)$ ) and  $\varepsilon > 0$  is given, then there exists a polynomial  $P$  such that  $\sup_{z \in K} |f(z) - P(z)| < \varepsilon$ .

The proof of the above theorem is not easy; however, if  $K = \overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ , then the proof is elementary: Since  $f$  is uniformly continuous on  $\overline{D}$ , there exists  $r, 0 < r < 1$ , so that  $|f(z) - f(rz)| < \frac{\varepsilon}{2}$ , for all  $z \in \overline{D}$ . On  $|w| \leq r$ , the convergence of the Taylor expansion of  $f$  towards  $f$  is uniform. Thus, there exists  $N \in \mathbb{N}$  so that  $|\sum_{n=0}^N \frac{f^{(n)}(0)}{n!} (rz)^n - f(rz)| < \frac{\varepsilon}{2}$  for all  $z \in \overline{D}$ . The triangle inequality implies  $|f(z) - P(z)| < \varepsilon$  for all  $z \in \overline{D}$ , where  $P$  is the polynomial  $P(z) = \sum_{n=0}^N [\frac{f^{(n)}(0)}{n!}] r^n z^n$ , which completes the proof.

In the above consideration we have uniform approximation on  $\overline{D}$  (or on  $K$ ) by polynomials with respect to the Euclidean metric on  $\mathbb{R}^2 = \mathbb{C}$ . However, if we consider functions taking the value  $\infty$ , as it happens very often in Complex Analysis, then the Euclidean metric on  $\mathbb{R}^2 = \mathbb{C}$ , does not allow us to speak about uniform approximation. But if we consider the one-point compactification  $\mathbb{C} \cup \{\infty\}$  of  $\mathbb{C}$  and the chordal distance  $\chi$  on it [1], then we can speak about uniform approximation on a compact set (or on several compact sets) of functions taking values in  $\mathbb{C} \cup \{\infty\}$ . Indeed, since  $\mathbb{C} \cup \{\infty\}$  is compact, any other metric giving the same topology is uniformly equivalent to the metric  $\chi$ . Thus, most of our results remain valid if we replace  $\chi$  by another equivalent metric on  $\mathbb{C} \cup \{\infty\}$ .

If we consider  $\Omega$  a simply connected domain in  $\mathbb{C}$ , then the uniform limits on compact subsets of  $\Omega$  of the polynomials with respect to the Euclidean metric on  $\mathbb{R}^2 = \mathbb{C}$ , are exactly all holomorphic functions  $f$  on  $\Omega$ . If we replace the Euclidean metric by the chordal metric  $\chi$ , then the limits are the same with an additional limit  $f \equiv \infty$ . Thus, one gets the impression that there is no essential difference.

Gauthier, Roth and Walsh in [4] compared two approximation theorems on a compact set  $K \subset \mathbb{C}$ ; one with respect to the Euclidean metric and the other with respect to the chordal metric  $\chi$ . It turned out that both theorems were valid exactly for the same compact sets  $K$ . Thus, again one gets the impression that there is no essential difference.

In [7,8], the uniform limits of the polynomials on  $\overline{D}$  (or on  $K$ ) with respect to the chordal distance  $\chi$  are investigated and compared with those defined with respect to the Euclidean metric. The limits with respect to the chordal metric are of two types: the infinite type, which contains only one function  $f \equiv \infty$  and the finite type. The finite type contains all holomorphic functions  $f : D \rightarrow \mathbb{C}$  on the open unit disc  $D$ , such that, for every  $\zeta \in \partial D$  the limit  $\lim_{z \rightarrow \zeta, z \in D} f(z)$  exists in  $\mathbb{C} \cup \{\infty\}$  (see also [2]).

All these functions are uniform limits of polynomials on  $\overline{D}$  with respect to  $\chi$  and only these functions. The set of these functions is denoted by  $\tilde{A}(D)$ .

The function  $\frac{1}{1-z}$  belongs to  $\tilde{A}(D)$  but not to the classical  $A(D)$ .

In [7,8], some properties of the elements of  $\tilde{A}(D)$  are investigated. Privalov's theorem implies that either the set  $(\partial D) \cap f^{-1}(\infty)$  is a compact set with zero length or  $f \equiv \infty$ . Conversely, every compact set  $K \subset \partial D$  with zero length coincides with  $f^{-1}(\infty)$  for an element  $f \in \tilde{A}(D)$  of finite type.

A natural question is to search for a characterization of all compact sets  $K \subset \partial D$  which are compact sets of interpolation for  $\tilde{A}(D)$ , that is, for all continuous functions  $\varphi : K \rightarrow \mathbb{C} \cup \{\infty\}$ , is it true that there exists  $f \in \tilde{A}(D)$  such that  $f|_K = \varphi$ . If  $K \subset \partial D$  has positive length, then we know that  $K$  is not a compact set of interpolation for  $\tilde{A}(D)$ . We do not know if every compact set  $K \subset \partial D$  with zero length is a compact set of interpolation of  $\tilde{A}(D)$ .

It is also true that for  $f, g \in \tilde{A}(D)$  the quantity  $\sup_{z \in \overline{D}} \chi(f(z), g(z))$  is not controlled by  $\sup_{\zeta \in \partial D} \chi(f(\zeta), g(\zeta))$ . Thus, the maximum principle fails. However, if  $f(\zeta) = g(\zeta)$  for all  $\zeta \in \partial(D)$  (or for a compact set of positive length in  $\partial D$ ) for some  $f, g \in \tilde{A}(D)$  then  $f \equiv g$ .

If  $f$  and  $g$  belong to  $\tilde{A}(D)$ , then the natural distance between  $f$  and  $g$  is  $\chi(f, g) = \sup_{z \in \overline{D}} \chi(f(z), g(z))$ . With this metric  $\tilde{A}(D)$  is a complete metric space. The classical disc algebra  $A(D)$  is an open and dense subset of  $\tilde{A}(D)$ . Furthermore, the relative topology of  $A(D)$  from  $\tilde{A}(D)$  coincides with the usual topology of  $A(D)$ .

Generically for all  $f \in \tilde{A}(D)$  the length of  $\{\zeta \in \partial D : f(\zeta) \notin f(D)\}$  is zero [7,8] (see also [14]). We also consider the sets  $X = \{f \in \tilde{A}(D) : f(D) \subset f(\partial D)\}$  and  $Y = \{f \in \tilde{A}(D) : f(\partial D) = \mathbb{C} \cup \{\infty\}\}$ .

Then  $X, Y$  are closed, nowhere dense subsets of  $\tilde{A}(D)$ . The sets  $X$  and  $Y$  are non-empty, because  $Y \subset X$  and it is known that  $Y \neq \emptyset$  (see [2]).

Replacing  $\bar{D}$  by other compact sets  $L \subset \mathbb{C}$  with  $L^c$  connected we arrive to the following:

**Question.** Let  $L \subset \mathbb{C}$  be a compact set with  $L^c$  connected,  $f : L \rightarrow \mathbb{C} \cup \{\infty\}$  a continuous function such that, for every component  $V$  of  $L^c$  it holds  $f|_V \equiv \infty$  or  $f(V) \subset \mathbb{C}$  and  $f|_V$  holomorphic. Is it then true that there exists a sequence of polynomials  $P_n$  so that  $P_n \rightarrow f$  uniformly on  $L$  with respect to the metric  $\chi$ ?

We are able to answer in the affirmative the previous question in some particular cases. The general case remains open.

### 2. Another compactification

In [9,10]  $\mathbb{C}$  is identified with  $D$  by the homeomorphism  $\mathbb{C} \ni z \rightarrow \frac{z}{1+|z|} \in D$ . This way  $\bar{D}$  becomes an obvious compactification of  $D$  endowed with the usual Euclidean metric. This induces a compactification of  $\mathbb{C}$ , where there are several points at  $\infty$  situated on a circle:  $\bar{\mathbb{C}} = \mathbb{C} \cup \mathbb{C}^\infty$ ,  $\mathbb{C}^\infty = \{\infty e^{i\vartheta} : \vartheta \in [0, 2\pi)\}$ . The metric  $d$  on  $\bar{\mathbb{C}}$  is  $d(z, w) = |\frac{z}{1+|z|} - \frac{w}{1+|w|}|$ , for  $z, w \in \mathbb{C}$ ,  $d(z, \infty e^{i\vartheta}) = |\frac{z}{1+|z|} - e^{i\vartheta}|$  for  $z \in \mathbb{C}$ ,  $\vartheta \in \mathbb{R}$  and  $d(\infty e^{i\varphi}, \infty e^{i\vartheta}) = |e^{i\varphi} - e^{i\vartheta}|$ ,  $\varphi, \vartheta \in \mathbb{R}$ .

The set of uniform limits  $\bar{A}(D)$  of the polynomials on  $\bar{D}$  with respect to  $d$  consists of the finite type and the infinite type. The finite type contains all holomorphic functions  $f : D \rightarrow \mathbb{C}$  such that, for every  $\zeta \in \partial D$  the limit  $\lim_{z \rightarrow \zeta, z \in D} f(z)$  exists in  $\bar{\mathbb{C}} = \mathbb{C} \cup \mathbb{C}^\infty$ .

The infinite type contains all continuous functions  $f : \bar{D} \rightarrow \mathbb{C}^\infty$ ,  $f(z) = \infty e^{i\vartheta(z)}$ , where  $\vartheta : \bar{D} \rightarrow \mathbb{R}$  is continuous on  $\bar{D}$  and harmonic in  $D$ .

Furthermore, in [9,10] one finds an investigation of properties of the elements of  $\bar{A}(D)$ , topological properties of  $\bar{A}(D)$  endowed with its natural metric topology, as well as possible extensions of Mergelyan's theorem.

### 3. The general case

We consider an arbitrary metrizable compactification  $S = \mathbb{C} \cup \mathbb{C}^\infty$  of  $\mathbb{C}$  (then the compact plane  $\mathbb{C}$  is an open dense subset of  $S$ , because  $\mathbb{C}$  is locally compact, [3]). We denote the metric on  $S$  by  $\rho$ . Two such examples are given in Section 1 and Section 2. However, there are several other compactifications.

- (i) Let  $\Omega$  be a bounded simply connected domain in  $\mathbb{C}$ . By the Riemann mapping theorem  $\Omega$  is homeomorphic to  $\mathbb{C}$ . Then  $\bar{\Omega}$  endowed with the Euclidean distance in  $\mathbb{C}$  induces a metrizable compactification of  $\mathbb{C}$ .
- (ii) Let  $\Omega$  be an unbounded simply connected domain in  $\mathbb{C}$ . Then the closure of  $\Omega$  in  $\mathbb{C} \cup \{\infty\}$  endowed with the metric  $\chi$  induces a metrizable compactification of  $\mathbb{C}$ .
- (iii) A torus in  $\mathbb{R}^3$  endowed with the Euclidean metric from  $\mathbb{R}^3$  induces a metrizable compactification of  $\mathbb{C}$ ; because, if we cut the torus along two circles it becomes homeomorphic to an open rectangle which is homeomorphic to  $\mathbb{C}$ . More generally, any compact surface in  $\mathbb{R}^3$  gives a metrizable compactification of  $\mathbb{C}$ , because it becomes homeomorphic to  $\mathbb{C}$  if we cut it accordingly. We refer to [5] for a classification of the compact surfaces.
- (iv) The real projective space of dimension 2 is a metrizable compactification of  $\mathbb{C}$ . One model of it is closed unit disc  $\bar{D}$ , where all the points  $w, -w \in \partial D$  are identified. This compactification is metrizable [13].
- (v) We consider  $\Omega_1 = D$ ; then  $\bar{D}$  induces a compactification  $\mathbb{C} \cup \mathbb{C}^\infty$  of  $\mathbb{C}$  where  $\mathbb{C} \approx D$  and  $\mathbb{C}^\infty \approx \partial D$ . Next we consider  $\Omega_2 = D - [0, 1)$ ; then  $\bar{D} = \bar{\Omega}_2$  induces a compactification  $\mathbb{C} \cup \mathbb{C}'^\infty$  of  $\mathbb{C}$  where  $\mathbb{C} \approx D - [0, 1)$  and  $\mathbb{C}'^\infty \approx \partial D \cup [0, 1)$ . These two compactifications will be considered as different for our purposes.

So, we fix a metrizable compactification  $S = \mathbb{C} \cup \mathbb{C}^\infty$  endowed with a metric  $\rho$  and we denote by  $A(D, S)$  the set of uniform limits with respect to  $\rho$  of all polynomials on  $\bar{D}$ . Let  $\phi : S = \mathbb{C} \cup \mathbb{C}^\infty \rightarrow \mathbb{C} \cup \{\infty\}$  be the function  $\phi(w) = w$  for  $w \in \mathbb{C}$  and  $\phi(w) = \infty$  for  $w \in \mathbb{C}^\infty$ . It is easily seen that  $\phi$  is continuous. Since  $S$  is compact, it follows that  $\phi$  is uniformly continuous. Thus, if  $P_n \rightarrow f$  uniformly on  $\bar{D}$  with respect to  $\rho$ , it follows that  $P_n = \phi(P_n) \rightarrow \phi(f)$  uniformly on  $\bar{D}$  with respect to the metric  $\chi$ . (Here  $P_n$  are polynomials.) Combining this with the results of Section 1 we see that  $A(D, S)$  consists of two types: the finite type and the infinite type. The finite type contains all functions  $f : D \rightarrow \mathbb{C}$  holomorphic, such that  $\lim_{z \rightarrow \zeta, z \in D} f(z)$  exists in  $S$  for all  $\zeta \in \partial D$ . It is easily seen that every such function can be approximated by polynomials. The infinite type contains functions  $f : \bar{D} \rightarrow \mathbb{C}^\infty$  which are continuous; but it is not true in general that all such functions can be approximated by polynomials. In the example of Section 2, we see that only some of these functions can be approximated by polynomials. Thus, there is no general result. In each particular case one should investigate which exactly are the functions of the infinite type belonging to  $A(D, S)$ .

So, we denote by  $A^f(D, S)$  and  $A^{inf}(D, S)$  the sets of elements of  $A(D, S)$  which are of finite type and, respectively, of infinite type. We have  $A(D, S) \supset A^f(D, S) \supset A(D)$  and  $A^{inf}(D, S) \neq \emptyset$ . This holds because  $A(D, S)$  contains all the constants in  $S$ : If  $b \in \mathbb{C}$ , then  $A(D, S)$  contains the constant polynomial  $b$ . If  $b \in \mathbb{C}^\infty$ , let  $b_n \in \mathbb{C}$  with  $b_n \rightarrow b$  as  $n \rightarrow \infty$ . If  $f_{b_n}$  denotes the constant polynomial equal to  $b_n$ , then  $f_{b_n} \rightarrow f_b$  uniformly on  $\bar{D}$ , where  $f_b(z) = b$  for all  $z \in \bar{D}$ . Thus,  $A^{inf}(D, S)$  contains  $f_b$  and therefore it is non-void. Now, for  $f, g \in A(D, S)$  the natural metric is  $\rho(f, g) := \sup_{z \in \bar{D}} \rho(f(z), g(z))$ . Thus,  $A(D, S)$  becomes a complete metric space.  $A(D)$  with its natural topology is an open and dense subset of  $A(D, S)$ . The set  $A^f(D, S)$  is also open and dense in  $A(D, S)$ .

**Proposition 3.1.** *Let  $S$  and  $S'$  be two metrizable compactifications of  $\mathbb{C}$  as above. Then the following are equivalent:*

- (1) *There exists a homeomorphism  $T : S = \mathbb{C} \cup \mathbb{C}^\infty \rightarrow S' = \mathbb{C} \cup \mathbb{C}'^\infty$  so that  $T(w) = w$  for all  $w \in \mathbb{C}$ .*
- (2) *There exists a homeomorphism  $F : A(D, S) \rightarrow A(D, S')$  so that for every polynomial  $P$  we have  $F(P) = P$ .*

One wonders if the condition  $A^f(D, S) = A^f(D, S')$  implies (1) of the previous proposition.

The answer is negative. Let  $S$  be the compactification where  $\mathbb{C} \approx D$  and  $\mathbb{C}^\infty = \partial D$  (the example in Section 2). Let  $S'$  be the example of the real projective space of dimension 2; that is,  $\mathbb{C} \approx D$  again but  $\mathbb{C}'^\infty$  is the quotient of  $\partial D$  where all points  $w$  and  $-w$  in  $\partial D$  are identified. Let  $f : D \rightarrow \mathbb{C}$  holomorphic. Let  $\zeta \in \partial D$ . Suppose  $\lim_{z \rightarrow \zeta, z \in D} f(z)$  exists in  $\mathbb{C}'^\infty$ . Then for a connected open subset  $V$  of  $D$  accumulating in  $\zeta$ , the set  $f(V)$  is contained in the union of two disjoint connected open sets one accumulating at  $w$  and the other at  $-w$ . Since by the continuity of  $f$ , the set  $f(V)$  should be connected, it follows that  $f(V)$  is either close to  $w$  or to  $-w$ . Thus,  $\lim_{z \in \zeta, z \in D} f(z)$  exists in  $\mathbb{C}^\infty$ . This shows that  $A^f(D, S) = A^f(D, S')$  but  $S \neq S'$  in the sense of Proposition 3.1. We also note that  $A^f(D, \mathbb{C} \cup \{\infty\})$  contain  $A^f(D, S)$ , for all metrizable compactifications  $S$ .

An open question is the following:

**Question.** Is there a metrizable compactification  $S$  of  $\mathbb{C}$  so that  $A^f(D, S) = A(D)$ ?

Another question relating to Mergelyan's theorem is the following:

**Question.** Let  $L \subset \mathbb{C}$  be a compact set with  $L^c$  connected and  $f : L \rightarrow S$  continuous, such that  $f(L^0) \subset \mathbb{C}$  and  $f|_{L^0}$  is holomorphic. Does there exist a sequence of polynomials  $P_n$  such that  $P_n \rightarrow f$  uniformly on  $L$  with respect to the metric  $\rho$  of  $S$ ?

In the special case where  $L$  is the closure of a Jordan domain  $\Omega$  in  $\mathbb{C}$ , the answer to the above question is affirmative (see also [11]). The reason is that if  $\varphi : D \rightarrow \Omega$  is a Riemann map then, according to a theorem of Caratheodory [6],  $\varphi$  extends to a homeomorphism  $\varphi : \overline{D} \rightarrow \overline{\Omega}$ . Thus, we are reduced to the case  $L = \overline{D}$ , where the result is known. We notice that in general the composition of a Riemann map with a polynomial is not a polynomial. However, it can be approximated by polynomials according to the classical Mergelyan's theorem. This is possible because  $\overline{\Omega}^c$  is connected for any Jordan domain  $\Omega$ .

## Acknowledgements

We would like to thank Richard Fournier for his contribution towards improving this note. Also we would like to thank the anonymous referee, especially for bringing to our attention Ref. [2].

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