C. R. Acad. Sci. Paris, Ser. I 349 (2011) 735–739

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Group Theory/Algebraic Geometry

On geometric properties of orbital varieties in type A

Sur des propriétés géométriques des variétés orbitales pour le type A

Lucas Fresse [a,1], Anna Melnikov [b]

[a] *Einstein Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel*
[b] *Department of Mathematics, University of Haifa, Haifa 31905, Israel*

ARTICLE INFO

Article history:
Received 11 November 2010
Accepted 31 May 2011
Available online 22 June 2011

Presented by the Editorial Board

ABSTRACT

The intersection between a nilpotent orbit $\mathcal{O} \subset \mathfrak{gl}_n(\mathbb{C})$ and the Lie algebra $Lie(B) \subset \mathfrak{gl}_n(\mathbb{C})$ of a Borel subgroup $B \subset GL_n(\mathbb{C})$ is an equidimensional, quasi-affine algebraic variety. Its irreducible components are called orbital varieties. In this Note, we provide criteria to guarantee that an orbital variety is smooth or has a dense orbit for the adjoint action of B. In addition, we point out a possible relation between these two properties.

RÉSUMÉ

L'intersection entre une orbite nilpotente $\mathcal{O} \subset \mathfrak{gl}_n(\mathbb{C})$ et l'algèbre de Lie $Lie(B) \subset \mathfrak{gl}_n(\mathbb{C})$ d'un sous-groupe de Borel $B \subset GL_n(\mathbb{C})$ est une variété algébrique quasi-affine équidimensionnelle. Ses composantes irréductibles sont appelées variétés orbitales. Dans cette Note, on propose des critères pour qu'une variété orbitale soit lisse ou bien possède une orbite dense pour l'action adjointe de B. De plus, on souligne un lien possible entre ces deux propriétés.

Version française abrégée

On note $G = GL_n(\mathbb{C})$ le groupe des matrices $n \times n$ inversibles et $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ l'espace des matrices $n \times n$, son algèbre de Lie. Soit $B \subset G$ le sous-groupe des matrices triangulaires supérieures et soit $\mathfrak{n} \subset \mathfrak{g}$ le sous-espace des matrices triangulaires supérieures de diagonale nulle. On note par $g \cdot x := gxg^{-1}$ l'action adjointe de G sur $\mathfrak{g}$. Une orbite $\mathcal{O}_x := G \cdot x$ $(x \in \mathfrak{g})$ est dite nilpotente si elle rencontre $\mathfrak{n}$. Dans ce cas, l'intersection $\mathcal{O}_x \cap \mathfrak{n}$ est une variété algébrique quasi-affine, équidimensionnelle, de dimension $\frac{1}{2} \dim \mathcal{O}_x$ (cf. [5]). Les composantes irréductibles de $\mathcal{O}_x \cap \mathfrak{n}$ sont appelées *variétés orbitales*. Les variétés orbitales interviennent en théorie géométrique des représentations, s'interprétant comme les analogues géométriques des idéaux primitifs de l'algèbre enveloppante $U(\mathfrak{g})$ (cf. [4]). Dans cette Note, on étudie des propriétés géométriques des variétés orbitales. Plus précisément, on étudie leur lissité et la propriété de posséder une orbite dense pour l'action adjointe de B.

Correspondance entre variétés orbitales et composantes des fibres de Springer

Pour $x \in \mathfrak{g}$ nilpotent, la *fibre de Springer* $\mathcal{B}_x$ est la variété des sous-groupes de Borel $B' \subset G$ tels que $x \in Lie(B')$. C'est une variété projective équidimensionnelle (cf. [5]).

E-mail addresses: lucasfresse@math.huji.ac.il (L. Fresse), melnikov@math.haifa.ac.il (A. Melnikov).
[1] Supported by the European project GLC, ERC grant No. 247049.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.05.016

Notons $G_x := \{g \in G\colon\ g^{-1} \cdot x \in \mathfrak{n}\}$. Les applications $\pi_1 : G_x \to \mathcal{O}_x \cap \mathfrak{n}$, $g \mapsto g^{-1} \cdot x$ et $\pi_2 : G_x \to \mathcal{B}_x$, $g \mapsto gBg^{-1}$ réalisent $\mathcal{O}_x \cap \mathfrak{n}$ et $\mathcal{B}_x$ comme les quotients de G_x par les actions de $Z_x := \{g \in G\colon\ g \cdot x = x\}$ (à gauche) et B (à droite), respectivement. On déduit que l'application $\Xi : X \mapsto \pi_2(\pi_1^{-1}(X))$ est une bijection entre les variétés orbitales $X \subset \mathcal{O}_x \cap \mathfrak{n}$ et les composantes irréductibles de $\mathcal{B}_x$. De plus, X est lisse (resp. admet une B-orbite dense) si et seulement si $\Xi(X)$ est lisse (resp. admet une Z_x-orbite dense).

Variétés orbitales paramétrées par des tableaux de Young standards

Soit $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_r)$ la suite des blocs de Jordan de x, et on note $\mathcal{O}_\lambda := \mathcal{O}_x$. Soit Y_λ le diagramme de Young de lignes de longueurs $\lambda_1, \ldots, \lambda_r$. Soit T un tableau de Young standard de forme Y_λ. Pour $1 \leqslant i < j \leqslant n$, on note $i <_T j$ si j est situé dans une colonne strictement à droite de la colonne de i dans T. Soit $(e_{i,j})_{1 \leqslant i < j \leqslant n}$ la base canonique de $\mathfrak{n}$. L'application $T \mapsto \mathcal{V}_T := \overline{B \cdot (\bigoplus_{i <_T j} \mathbb{C} e_{i,j})} \cap \mathcal{O}_\lambda$ est une bijection entre les tableaux standards de forme Y_λ et les variétés orbitales de $\mathcal{O}_\lambda$ (cf. [6]).

On note par $\lambda^* = (\lambda_1^* \geqslant \cdots \geqslant \lambda_{\lambda_1}^*)$ la suite des longueurs des colonnes de Y_λ. Soit T^* le tableau obtenu d'après T par transposition. Alors, $\mathcal{V}_T \mapsto \mathcal{V}_{T^*}$ est une bijection entre les variétés orbitales de $\mathcal{O}_\lambda$ et $\mathcal{O}_{\lambda^*}$. Cette bijection jouera un rôle dans nos résultats.

Variétés orbitales de Richardson et de Bala-Carter

Une variété orbitale $\mathcal{V}$ est dite de *Richardson* si son adhérence $\overline{\mathcal{V}}$ est le nilradical d'une sous-algèbre parabolique $\mathfrak{p} \subset \mathfrak{g}$ standard (i.e. telle que $\mathfrak{p} \supset Lie(B)$). Une variété orbitale de Richardson est donc toujours lisse. D'autre part, $\mathcal{V}$ est dite de *Bala-Carter* si $\mathcal{V}$ contient un élément régulier y d'un facteur de Levi $\mathfrak{l}$ d'une sous-algèbre parabolique standard. L'orbite $B \cdot y$ est alors dense dans $\mathcal{V}$. Une variété orbitale de Bala-Carter possède donc toujours une B-orbite dense. Chaque orbite nilpotente $\mathcal{O} \subset \mathfrak{gl}_n(\mathbb{C})$ contient au moins une variété orbitale de Richardson (resp. de Bala-Carter). On a par ailleurs la relation suivante : $\mathcal{V}_T$ est de Richardson si et seulement si $\mathcal{V}_{T^*}$ est de Bala-Carter.

Résultats principaux

Dans [1–3], nous avons étudié la lissité des composantes des fibres de Springer. Avec l'aide de la correspondance décrite ci-dessus, nous déduisons les trois résultats suivants pour la lissité des variétés orbitales :

- une description des orbites nilpotentes dont toutes les variétés orbitales sont lisses : toutes les variétés orbitales de $\mathcal{O}_\lambda$ sont lisses dans exactement quatre cas : (i) $\lambda = (\lambda_1, 1, \ldots, 1)$ (Y_λ de type « crochet »); (ii) $\lambda = (\lambda_1, \lambda_2)$ (i.e. Y_λ a deux lignes); (iii) $\lambda = (\lambda_1, \lambda_2, 1)$; (iv) $\lambda = (2, 2, 2)$,
- un critère nécessaire et suffisant de lissité pour $\mathcal{V}_T$ lorsque le tableau T a deux colonnes,
- un critère nécessaire et suffisant de lissité pour les variétés orbitales de type Bala-Carter.

Les énoncés précis figurent dans la Section 5.

À présent, nous avons étudié l'existence de B-orbite dense pour les variétés orbitales. Les critères obtenus s'avèrent très similaires aux critères de lissité précédents. Ils peuvent s'énoncer sous la forme du théorème suivant, qui suggère un lien entre lissité et existence de B-orbite dense pour les variétés orbitales :

Théorème 0.1. (a) *Toute variété orbitale contenue dans l'orbite nilpotente $\mathcal{O}_\lambda$ a une B-orbite dense si et seulement si toute variété orbitale contenue dans l'orbite $\mathcal{O}_{\lambda^*}$ est lisse.*

(b) *Soit T un tableau standard à deux lignes (resp. soit $\mathcal{V}_T$ une variété orbitale de type Richardson). Alors, $\mathcal{V}_T$ a une B-orbite dense si et seulement si $\mathcal{V}_{T^*}$ est lisse.*

1. Introduction

Let $G = GL_n(\mathbb{C})$ be the group of $n \times n$ invertible matrices, $B \subset G$ the subgroup of upper triangular matrices, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ the space of $n \times n$ matrices, and $\mathfrak{n} \subset \mathfrak{g}$ the subspace of strictly upper triangular matrices. The group G operates on $\mathfrak{g}$ by the adjoint action $(g, x) \mapsto g \cdot x := gxg^{-1}$. An adjoint orbit $\mathcal{O}_x := G \cdot x$ ($x \in \mathfrak{g}$) is called nilpotent if it intersects $\mathfrak{n}$. Then, the intersection $\mathcal{O}_x \cap \mathfrak{n}$ is a quasi-affine algebraic variety, of pure dimension $\frac{1}{2} \dim \mathcal{O}_x$ (cf. [5]). The irreducible components of $\mathcal{O}_x \cap \mathfrak{n}$ are called *orbital varieties*. Orbital varieties arise in geometric representation theory where they interpret as the geometric analogues of the primitive ideals of the enveloping algebra $U(\mathfrak{g})$ (cf. [4]).

However, orbital varieties can be complicated geometric objects. In general, they are not smooth and, though they are stabilized by the adjoint action of B, they have in general no dense B-orbit. In this Note, we give criteria for an orbital variety to be smooth or to have a dense B-orbit. Our results suggest a possible relation between these two properties.

2. Orbital varieties and Springer fiber components

To study geometric properties of orbital varieties, it is useful to rely on Springer fibers.

The *flag variety* $\mathcal{B} = G/B$ is the variety of Borel subgroups $B' \subset G$. Given a nilpotent element $x \in \mathfrak{g}$, the *Springer fiber* $\mathcal{B}_x$ is the subvariety of Borel subgroups $B' \subset G$ such that $x \in Lie(B')$. In fact, $\mathcal{B}_x$ is the fiber over x of the Springer resolution of the nilcone of $\mathfrak{g}$. The variety $\mathcal{B}_x$ is connected, projective and equidimensional (cf. [5]). Note that $\mathcal{B}$ interprets as the variety of complete flags $(V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n)$ with $\dim V_i = i$ for all i, and $\mathcal{B}_x$ interprets as the subvariety of x-stable flags, i.e. such that $x(V_i) \subset V_i$ for all i (seeing x as a nilpotent endomorphism of $\mathbb{C}^n$).

There is the following close relation between the orbital varieties of the nilpotent orbit $\mathcal{O}_x$ and the irreducible components of the Springer fiber $\mathcal{B}_x$. Consider the actions of B and $Z_x := \{g \in G\colon\ g \cdot x = x\}$ on G by right and left multiplication, respectively. The maps $\pi_1 : G \to \mathcal{O}_x \cong Z_x\backslash G,\ g \mapsto g^{-1} \cdot x$ and $\pi_2 : G \to \mathcal{B} = G/B,\ g \mapsto gBg^{-1}$ are the corresponding quotient maps for these actions, hence they are open, smooth and surjective. Note the equality $\pi_1^{-1}(\mathcal{O}_x \cap \mathfrak{n}) = \pi_2^{-1}(\mathcal{B}_x)$. Since moreover Z_x and B are connected subgroups of G, it easily follows:

Proposition 2.1. *The mapping $\Xi : X \mapsto \pi_2(\pi_1^{-1}(X))$ is a bijection between orbital varieties of $\mathcal{O}_x$ and irreducible components of the Springer fiber $\mathcal{B}_x$. Furthermore, X is smooth if and only if $\Xi(X)$ is smooth, and X has a dense B-orbit if and only if $\Xi(X)$ has a dense Z_x-orbit.*

Thereby, studying the smoothness and the existence of dense B-orbits for orbital varieties is equivalent to studying the smoothness and the existence of dense Z_x-orbits for Springer fiber components.

3. Parameterization of orbital varieties by standard Young tableaux

A nilpotent element $x \in \mathfrak{g}$ is determined up to its nilpotent orbit by the sequence of the lengths of its Jordan blocks $\lambda(x) = (\lambda_1 \geqslant \cdots \geqslant \lambda_r)$, so that we can write $\mathcal{O}_\lambda := \mathcal{O}_x$ for $\lambda = \lambda(x)$. Alternatively, we represent λ by the Young diagram Y_λ with r rows of sizes $\lambda_1, \ldots, \lambda_r$ respectively. By λ^*, we denote the sequence $\lambda^* = (\lambda_1^* \geqslant \cdots \geqslant \lambda_{\lambda_1}^*)$ such that $\lambda_j^* = |\{i = 1, \ldots, r\colon\ \lambda_i \geqslant j\}|$. Thus, λ^* is the list of the lengths of the columns of Y_λ, and the corresponding Young diagram Y_{λ^*} is the transpose of Y_λ. We have $\dim \mathcal{O}_x \cap \mathfrak{n} = \frac{1}{2}(n^2 - \sum_{j=1}^{\lambda_1} (\lambda_j^*)^2)$ and $\dim \mathcal{B}_x = \sum_{j=1}^{\lambda_1} \frac{\lambda_j^*(\lambda_j^*-1)}{2}$.

We let $Tab(Y_\lambda)$ be the set of standard Young tableaux of shape Y_λ, i.e. numberings of Y_λ by $1, \ldots, n$ with numbers increasing from left to right along rows and from top to bottom along columns. The orbital varieties of $\mathcal{O}_\lambda$ have the following parameterization by tableaux $T \in Tab(Y_\lambda)$ (cf. [6]). For $1 \leqslant i < j \leqslant n$, we write $i <_T j$ if j is situated in a column strictly on the right of the column of i in the tableau T. We denote by $e_{i,j} \in \mathfrak{n}$ the elementary matrix with 1 at position (i, j) and 0 elsewhere, and we set $\mathfrak{g}_{i,j} = \mathbb{C}e_{i,j}$.

Proposition 3.1. *The mapping $T \mapsto \mathcal{V}_T := \overline{B \cdot (\bigoplus_{i <_T j} \mathfrak{g}_{i,j})} \cap \mathcal{O}_\lambda$ is a bijection from $Tab(Y_\lambda)$ to the set of orbital varieties of the nilpotent orbit $\mathcal{O}_\lambda$.*

Letting $\mathcal{K}^T = \Xi(\mathcal{V}_T)$, we get a construction of the components of the Springer fiber $\mathcal{B}_x$ (for $\lambda(x) = \lambda$), which coincides with Spaltenstein's classical construction (cf. [5, §II.5]).

Given a tableau $T \in Tab(Y_\lambda)$, we let $T^* \in Tab(Y_{\lambda^*})$ be the transposed tableau.

$$\text{For example,}\quad T = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 4&6\\\cline{1-2} 7&8\\\cline{1-2}\end{array} \quad\Rightarrow\quad T^* = \begin{array}{|c|c|c|}\hline 1&4&7\\\hline 2&6&8\\\hline 5\\\cline{1-1}\end{array}.$$

The mapping $\mathcal{V}_T \mapsto \mathcal{V}_{T^*}$ provides a one-to-one correspondence between orbital varieties of the nilpotent orbits $\mathcal{O}_\lambda$ and $\mathcal{O}_{\lambda^*}$. This correspondence will play a central role in our results.

4. Richardson and Bala-Carter orbital varieties

We describe now two special families of orbital varieties. Let $\pi = (\pi_1, \ldots, \pi_r)$ be a sequence of positive integers, obtained by permuting the terms in the sequence $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_r)$. Let $\mathfrak{l}_\pi$ (resp. $\mathfrak{p}_\pi$) (resp. $\mathfrak{n}_\pi$) $\subset \mathfrak{g}$ be the subspace of block-wise diagonal (resp. upper triangular) (resp. strictly upper triangular) matrices with blocks of sizes $\pi_1, \ldots, \pi_r$ along the diagonal. Thus, $\mathfrak{p}_\pi$ is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_\pi = \mathfrak{l}_\pi \oplus \mathfrak{n}_\pi$ is a Levi decomposition.

The nilpotent orbit $\mathcal{O}_{\lambda^*}$ intersects the nilradical $\mathfrak{n}_\pi$ densely, moreover it turns out that $\mathcal{V}_\pi^R := \mathcal{O}_{\lambda^*} \cap \mathfrak{n}_\pi$ is an orbital variety of $\mathcal{O}_{\lambda^*}$ (cf. [6, §4]). We call $\mathcal{V}_\pi^R$ a *Richardson orbital variety*. It is smooth by construction.

The nilpotent orbit $\mathcal{O}_\lambda$ contains the regular nilpotent orbit of $\mathfrak{l}_\pi$. Taking $y \in \mathfrak{l}_\pi \cap \mathfrak{n}$ regular, it turns out that $\mathcal{V}_\pi^{BC} := \overline{B \cdot y} \cap \mathcal{O}_\lambda$ is an orbital variety of $\mathcal{O}_\lambda$ (independent of the choice of y, cf. [6, §4]). We call $\mathcal{V}_\pi^{BC}$ a Bala-Carter orbital variety. It contains a dense B-orbit by construction.

Richardson and Bala-Carter orbital varieties are characterized as follows in terms of tableaux. Draw an array with r left-adjusted rows of lengths $\pi_1, \ldots, \pi_r$. Number the first row by $1, \ldots, \pi_1$, the second row by $\pi_1 + 1, \ldots, \pi_1 + \pi_2$, etc. Then, push to the top the supernumerary boxes in each row so to obtain a standard tableau, which we call T_π^{BC}.

$$\text{For example,} \quad \pi = (2, 1, 4) \quad \Rightarrow \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & \multicolumn{3}{c}{} \\ \hline 4 & 5 & 6 & 7 \\ \hline \end{array} \quad \Rightarrow \quad T_\pi^{BC} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 6 & 7 \\ \hline 3 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}.$$

Let $T_\pi^R = (T_\pi^{BC})^*$ be the transpose of T_π^{BC}. Then, we have $\mathcal{V}_\pi^{BC} = \mathcal{V}_{T_\pi^{BC}}$ and $\mathcal{V}_\pi^R = \mathcal{V}_{T_\pi^R}$.

5. On smooth orbital varieties

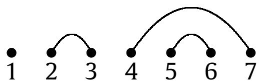
In [1–3], we obtained results on the singularity of Springer fiber components, that we can now translate, thanks to Proposition 2.1, into results for orbital varieties. Note first that each nilpotent orbit $\mathcal{O}_\lambda$ contains at least one Richardson orbital variety, hence at least one smooth orbital variety. In general, however, not all the orbital varieties of $\mathcal{O}_\lambda$ are smooth:

Theorem 5.1. *Let $\mathcal{O}_\lambda \subset \mathfrak{gl}_n(\mathbb{C})$ be a nilpotent orbit corresponding to the sequence $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_r)$. There are exactly four cases where all the orbital varieties of $\mathcal{O}_\lambda$ are smooth*:

(i) *the "hook case"*: $\lambda_2 = \cdots = \lambda_r = 1$;
(ii) *the "two-row case"*: $\lambda = (\lambda_1, \lambda_2)$;
(iii) *the "two-row-plus-one-box" case*: $\lambda = (\lambda_1, \lambda_2, 1)$;
(iv) *the case* $\lambda = (2, 2, 2)$.

The names of the cases refer to the form taken by the diagram Y_λ. The case of $\mathcal{O}_\lambda$ where $\lambda_i \leqslant 2$ for all i is called "two-column case" and in general singular orbital varieties arise in this case. Our second result is a criterion of singularity for orbital varieties in the two-column case. Given $T \in Tab(Y_\lambda)$ with two columns, the corresponding *cup-diagram* P_T is the graph with n vertices labeled by $1, \ldots, n$ and displayed along a horizontal line, and edges constructed as follows. We run over the set of numbers $i_1, \ldots, i_{\lambda_2^*}$ in the second column of T, and for each j we draw an arc from the vertex i_j to the closest vertex on its left which is not yet a point of an arc.

$$\text{For example,} \quad T = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 & 7 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \Rightarrow \quad P_T =$$



A vertex i is called an end point if an arc starts or ends at i. Let $\tau^*(T) = \{i\colon\ (i, i+1) \text{ is an arc of } P_T\}$.

Theorem 5.2. *Let T be a standard Young tableau with two columns. The orbital variety $\mathcal{V}_T$ is smooth exactly in the following three cases*:

(i) $|\tau^*(T)| = 1$;
(ii) $|\tau^*(T)| = 2$, *and* 1 *or* n *is an end point of* P_T;
(iii) $|\tau^*(T)| = 3$, *and* 1 *and* n *are both end points of* P_T, *but not of the same arc.*

For instance, the tableau T drawn above fits in case (ii) of the theorem, hence $\mathcal{V}_T$ is smooth.

Finally, we give a criterion of singularity for Bala-Carter orbital varieties. Given two sequences $\pi = (\pi_1, \ldots, \pi_r)$ and $\rho = (\rho_1, \ldots, \rho_s)$, write $\pi \geqslant \rho$ if there are $1 \leqslant i_1 < \cdots < i_s \leqslant r$ with $\pi_{i_j} \geqslant \rho_j$ for all j.

Theorem 5.3. *Let $\pi = (\pi_1, \ldots, \pi_r)$ be a sequence of positive integers. The Bala-Carter orbital variety $\mathcal{V}_\pi^{BC}$ is singular if and only if $\pi \geqslant (1, 2, 2, 1)$ or $\pi \geqslant (2, 3, 2)$.*

Theorem 5.3 shows that Bala-Carter orbital varieties are the most susceptible to be singular. Note that Theorem 5.1 means that $\mathcal{O}_\lambda$ admits a singular orbital variety exactly when $\lambda \geqslant (2, 2, 1, 1)$ or $\lambda \geqslant (3, 2, 2)$. Then, we see that $\mathcal{O}_\lambda$ always admits singular Bala-Carter orbital varieties in this case. Further, if $\lambda \geqslant (2, 2, 2, 2)$ or $\lambda \geqslant (3, 3, 3)$, then all the Bala-Carter orbital varieties of $\mathcal{O}_\lambda$ are singular.

Idea of proof. The combination of the following properties is sufficient to generate all the three results:

- Let T' be obtained from T by deleting the box number n (the maximal number of T). If $\mathcal{V}_{T'}$ is singular, then $\mathcal{V}_T$ is singular, and equivalence holds if n lies in the last column of T.
- $\mathcal{V}_T$ is smooth if and only if $\mathcal{V}_{Sch(T)}$ is smooth, where $T \mapsto Sch(T)$ is the Schützenberger involution.

- If T is of hook type, then $\mathcal{V}_T$ is smooth.
- If T has two columns and $|\tau^*(T)| = 1$, then $\mathcal{V}_T$ is smooth.
- If T has two columns, $|\tau^*(T)| = 2$ and $1, n$ are not end points of P_T, then $\mathcal{V}_T$ is singular.
- For π a sequence of the form $\pi = (1, p, 1^s, q, 1)$ or $\pi = (2, 1^s, r, 1^t, 2)$, with $p, q \geqslant 2$, $r \geqslant 3$, $s, t \geqslant 0$ (where 1^t indicates a sequence of t 1's), the orbital variety $\mathcal{V}_\pi^{BC}$ is singular.
- For $\pi = (p, 1^s, q, 1^t)$, with $p, q, s, t \geqslant 1$, the orbital variety $\mathcal{V}_\pi^{BC}$ is smooth. □

6. On orbital varieties with a dense *B*-orbit

We analyze the question of existence of dense B-orbits in orbital varieties following the same program as in the previous section for smoothness. Observe that each nilpotent orbit $\mathcal{O}_\lambda \subset \mathfrak{gl}_n(\mathbb{C})$ admits at least one Bala-Carter orbital variety, which has a dense B-orbit. Our first result shows that, in most of the cases, $\mathcal{O}_\lambda$ admits orbital varieties with no dense B-orbit:

Theorem 6.1. *Let $\mathcal{O}_\lambda \subset \mathfrak{gl}_n(\mathbb{C})$ be the nilpotent orbit associated to the sequence $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_r)$. There are exactly four cases where all the orbital varieties of $\mathcal{O}_\lambda$ have a dense B-orbit*:

(i) *the hook case, i.e.* $\lambda_2 = \cdots = \lambda_r = 1$;
(ii) *the two-column case, i.e.* $\lambda^* = (\lambda_1^*, \lambda_2^*)$;
(iii) *the "two-column-plus-one-box" case*: $\lambda^* = (\lambda_1^*, \lambda_2^*, 1)$;
(iv) *the case* $\lambda = (3, 3)$.

Comparing Theorems 5.1 and 6.1 leads to the equivalence: all orbital varieties of $\mathcal{O}_\lambda$ have a dense B-orbit if and only if all orbital varieties of $\mathcal{O}_{\lambda^*}$ are smooth. Next, we give a criterion of existence of dense B-orbit for two-row type and Richardson orbital varieties, which we can formulate as follows:

Theorem 6.2. (a) *Let T be a standard tableau with two rows. Then, $\mathcal{V}_T$ has a dense B-orbit if and only if the orbital variety $\mathcal{V}_{T^*}$, associated to the transposed tableau T^*, is smooth.*

(b) *Let $T = T_\pi^R$ for a sequence $\pi = (\pi_1, \ldots, \pi_r)$, so that $\mathcal{V}_T = \mathcal{V}_\pi^R$ is a Richardson orbital variety. Then, $\mathcal{V}_T$ has a dense B-orbit if and only if the Bala-Carter orbital variety $\mathcal{V}_{T^*} = \mathcal{V}_\pi^{BC}$ is smooth.*

Idea of proof. The theorems are obtained by establishing the counterparts of the seven properties that we combined to get the theorems in the previous section, namely:

- Let T' be obtained from T by deleting the maximal entry n. If $\mathcal{V}_{T'}$ has no dense B-orbit, then $\mathcal{V}_T$ has no dense B-orbit, and equivalence holds if n lies in the last row of T.
- $\mathcal{V}_T$ has a dense B-orbit if and only if $\mathcal{V}_{Sch(T)}$ has a dense B-orbit.
- If T is of hook type, then $\mathcal{V}_T$ has a dense B-orbit.
- If T has two rows and $|\tau^*(T^*)| = 1$, then $\mathcal{V}_T$ has a dense B-orbit.
- If T has two rows, $|\tau^*(T^*)| = 2$ and $1, n$ are not end points of P_{T^*}, then $\mathcal{V}_T$ has no dense B-orbit.
- $\mathcal{V}_\pi^R$ has no dense B-orbit for $\pi = (1, p, 1^s, q, 1)$ or $\pi = (2, 1^s, r, 1^t, 2)$, with $p, q \geqslant 2$, $r \geqslant 3$, $s, t \geqslant 0$.
- $\mathcal{V}_\pi^R$ has a dense B-orbit for $\pi = (p, 1^s, q, 1^t)$, with $p, q, s, t \geqslant 1$. □

Theorems 6.1 and 6.2 suggest to conjecture that the tableau transposition $T \mapsto T^*$ provides a general correspondence between smooth orbital varieties and orbital varieties with a dense B-orbit. At this stage, we have no explanation for this unexpected phenomenon.

References

[1] L. Fresse, A. Melnikov, On the singularity of the irreducible components of a Springer fiber in $sl(n)$, Selecta Math. (N.S.) 16 (2010) 393–418.
[2] L. Fresse, A. Melnikov, Some characterizations of singular components of Springer fibers in the two-column case, Algebr. Rep. Theory, doi:10.1007/s10468-010-9227-5, in press.
[3] L. Fresse, On the singularity of some special components of Springer fibers, J. Lie Theory 21 (2011) 205–242.
[4] A. Joseph, On the variety of a highest weight module, J. Algebra 88 (1984) 238–278.
[5] N. Spaltenstein, Classes unipotentes et sous-groupes de Borel, Lecture Notes in Math., vol. 946, Springer-Verlag, 1982.
[6] R. Steinberg, On the desingularization of the unipotent variety, Invent. Math. 36 (1976) 209–224.