Probability Theory/Mathematical Physics

A deletion-invariance property for random measures satisfying the Ghirlanda–Guerra identities

Une propriété d'invariance-suppression des mesures aléatoires vérifiant les identités de Ghirlanda–Guerra

Dmitry Panchenko [1]

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA

A R T I C L E I N F O

A B S T R A C T

We show that if a discrete random measure on the unit ball of a separable Hilbert space satisfies the Ghirlanda–Guerra identities then by randomly deleting half of the points and renormalizing the weights of the remaining points we obtain the same random measure in distribution up to rotations.

© 2011 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Nous montrons que si une mesure aléatoire discrète sur la boule unité d'un espace de Hilbert séparable satisfait aux identités de Ghirlanda–Guerra, alors en supprimant aléatoirement la moitié des points et en renormalisant les poids des points restants, on obtient une mesure de même distribution à une rotation près.

© 2011 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction and main result

Let us consider a countable index set A and random probability measure μ on a unit ball B of a separable Hilbert space H such that $\mu = \sum_{\alpha \in A} w_\alpha \delta_{\xi_\alpha}$ for some random points $\xi_\alpha \in B$ and weights (w_α). We will call indices α from the set A "configuration" and for a function $f = f(\alpha^1, \ldots, \alpha^n)$ of n configurations we will denote its average with respect to the measure μ by

$$\langle f \rangle = \sum_{\alpha^1, \ldots, \alpha^n} w_{\alpha^1} \cdots w_{\alpha^n} f(\alpha^1, \ldots, \alpha^n). \tag{1}$$

We say that the random measure μ satisfies the Ghirlanda–Guerra identities [3] if for any $n \geqslant 2$ and any function f that depends on the configurations $\alpha^1, \ldots, \alpha^n$ only through the scalar products, or overlaps, $R_{\ell, \ell'} = \xi_{\alpha^\ell} \cdot \xi_{\alpha^{\ell'}}$ for $\ell, \ell' \leqslant n$ we have

$$\mathbb{E}\langle f R_{1,n+1}^p \rangle = \frac{1}{n}\mathbb{E}\langle f \rangle \mathbb{E}\langle R_{1,2}^p \rangle + \frac{1}{n}\sum_{\ell=2}^n \mathbb{E}\langle f R_{1,\ell}^p \rangle \tag{2}$$

for any integer $p \geqslant 1$. Random measures satisfying the Ghirlanda–Guerra identities arise as the directing measures (or *determinators* in the terminology of [8]) of overlap matrices in the Sherrington–Kirkpatrick model where they can be seen as the asymptotic analogues of the Gibbs measure. The importance of the asymptotic point of view provided by these measures was brought to light in [2], even though it was the Aizenman–Contucci stochastic stability [1] and not the Ghirlanda–Guerra identities that played the main role there. However, subsequently, such random measures satisfying the Ghirlanda–Guerra identities played an equally important role in the results of [4] and to a less extent of [6].

Our main result is based on a simple observation which extends the invariance theorem from [4]. Consider independent symmetric Bernoulli random variables $(\varepsilon_\alpha)_{\alpha \in A}$ (taking values ± 1 with probability $1/2$) and for $t \in \mathbb{R}$ let us define a random measure $\mu_t = \sum_{\alpha \in A} w_{\alpha,t} \delta_{\xi_\alpha}$ with weights defined by the random change of density

$$w_{\alpha,t} = \frac{w_\alpha \exp t \varepsilon_\alpha}{\sum_{\gamma \in A} w_\gamma \exp t \varepsilon_\gamma}, \tag{3}$$

and as in (1) let us denote the average with respect to this measure by

$$\langle f \rangle_t = \sum_{\alpha^1, \ldots, \alpha^n} w_{\alpha^1,t} \cdots w_{\alpha^n,t} f(\alpha^1, \ldots, \alpha^n). \tag{4}$$

The following holds:

Theorem 1.1. *If a random measure μ satisfies the Ghirlanda–Guerra identities (2) then for any $t \in \mathbb{R}$, any $n \geqslant 2$ and any bounded function f of the overlaps on n configurations we have $\mathbb{E}\langle f \rangle_t = \mathbb{E}\langle f \rangle$.*

The main difference here is that the result holds for all $t \in \mathbb{R}$ compared to $|t| < 1/2$ as stated in Theorem 4 in [4] which was sufficient for the main argument there. However, letting t go to infinity we now obtain the following new invariance property. Let $\eta_\alpha = (\varepsilon_\alpha + 1)/2$ be independent random variables, now taking values 1 and 0 with probability $1/2$ and let $\mu' = \sum_{\alpha \in A} w'_\alpha \delta_{\xi_\alpha}$ be the random measure defined by the change of density

$$w'_\alpha = \frac{w_\alpha \eta_\alpha}{\sum_{\gamma \in A} w_\gamma \eta_\gamma}. \tag{5}$$

In other words, we randomly delete half of the point in the support of measure μ and renormalize the weights to define a probability measure μ' on the remaining points. The denominator in (5) is non-zero with probability one since it is well known that unless the measure μ is concentrated at $0 \in B$ (a case we do not consider) it must have infinitely many different points in the support in order to satisfy (2). Let us define by $\langle f \rangle'$ the average with respect to μ'.

Theorem 1.2 (Deletion invariance). *If a random measure μ satisfies the Ghirlanda–Guerra identities (2) then $\mathbb{E}\langle f \rangle' = \mathbb{E}\langle f \rangle$.*

Remark 1. In particular, this implies that the measure μ' also satisfies the Ghirlanda–Guerra identities (2) and, hence, we can repeat the random deletion procedure as many times as we want. This means that the deletion invariance also holds with random variables (η_α) taking values 1 and 0 with probabilities $1/2^s$ and $1 - 1/2^s$ correspondingly, for any integer $s \geqslant 1$.

Remark 2. It is well known that invariance for the averages as in Theorem 1.2 implies that the random measures μ and μ' have the same distribution, up to rotations. Let $(w_\ell)_{\ell \geqslant 1}$ be the weights (w_α) arranged in the non-increasing order and let (ξ_ℓ) be the points (ξ_α) rearranged accordingly, so that $\mu = \sum_{\ell \geqslant 1} w_\ell \delta_{\xi_\ell}$. Similarly, let $\mu' = \sum_{\ell \geqslant 1} w'_\ell \delta_{\xi'_\ell}$. Then arguing as at the end of the proof of Theorem 4 in [4] (or Lemma 4 in [5]) one can show that

$$\big((w_\ell)_{\ell \geqslant 1}, (\xi_\ell \cdot \xi_{\ell'})_{\ell,\ell' \geqslant 1} \big) \overset{d}{=} \big((w'_\ell)_{\ell \geqslant 1}, (\xi'_\ell \cdot \xi'_{\ell'})_{\ell,\ell' \geqslant 1} \big) \tag{6}$$

which means that up to rotations the configurations of the random measures μ and μ' have the same distributions.

Proof of Theorem 1.1. Suppose that $|f| \leqslant 1$. Let $\varphi(t) = \mathbb{E}\langle f \rangle_t$ and by symmetry we only need to consider $t \geqslant 0$. Given configurations $\alpha^1, \alpha^2, \ldots$ let us denote

$$D_n = \varepsilon_{\alpha^1} + \cdots + \varepsilon_{\alpha^n} - n \varepsilon_{\alpha^{n+1}}$$

and a straightforward computation shows that $\varphi'(t) = \mathbb{E}\langle f D_n \rangle_t$ and similarly for all $k \geqslant 1$,

$$\varphi^{(k)}(t) = \mathbb{E}\langle f D_n \cdots D_{n+k-1} \rangle_t.$$

It was proved in Theorem 4 in [4] (a more streamlined proof was given in Theorem 6.3 in [6]) that if the measure μ satisfies the Ghirlanda–Guerra identities (2) then

$$\varphi^{(k)}(0) = 0 \quad \text{for all } k \geqslant 1. \tag{7}$$

It is also easy to see that $|D_n| \leqslant 2n$ so that for all t,

$$\left|\varphi^{(k)}(t)\right| \leqslant 2^k n(n+1)\cdots(n+k-1). \tag{8}$$

This is all one needs to show that if

$$\varphi(t) = \varphi(0) \quad \text{and} \quad \varphi^{(k)}(t) = 0 \quad \text{for all } k \geqslant 1 \tag{9}$$

holds for all $t \leqslant t_0$ for some $t_0 \geqslant 0$ then it also holds for all $t < t_0 + 1/2$. This will finish the proof of the theorem since by (7) this holds for $t_0 = 0$. Take any $k \geqslant 0$. Using (8) and (9) for $t = t_0$ and using Taylor's expansion for a function $\varphi^{(k)}(t)$ around the point $t = t_0$ we get for any $m \geqslant 1$

$$\left|\varphi^{(k)}(t) - \varphi^{(k)}(t_0)\right| \leqslant \sup_{t_0 \leqslant s \leqslant t} \frac{|\varphi^{(k+m)}(s)|}{m!} |t - t_0|^m \leqslant \frac{2^{k+m} n(n+1)\cdots(n+k+m-1)}{m!} |t - t_0|^m.$$

If $|t - t_0| < 1/2$ then letting $m \to \infty$ proves that $\varphi^{(k)}(t) = \varphi^{(k)}(t_0)$ for all $k \geqslant 0$ and therefore (9) holds for all $t < t_0 + 1/2$. $\quad\square$

Proof of Theorem 1.2. Let $I = \{\alpha \in A: \ \varepsilon_\alpha = 1\}$ and let

$$Z_t = \sum_{\alpha \in I} w_\alpha + e^{-2t} \sum_{\alpha \in I^c} w_\alpha$$

so that

$$w_{\alpha,t} = \frac{w_\alpha}{Z_t}\left(I(\alpha \in I) + e^{-2t} I(\alpha \in I^c)\right). \tag{10}$$

Then the sum on the right-hand side of (4) can be broken into 2^n groups depending on which of the indexes $\alpha^1, \ldots, \alpha^n$ belong to I or its complement I^c, for example, the terms corresponding to all indices belonging to I will give

$$\frac{1}{Z_t^n} \sum_{\alpha^1, \ldots, \alpha^n \in I} w_{\alpha^1} \cdots w_{\alpha^n} f\left(\alpha^1, \ldots, \alpha^n\right). \tag{11}$$

This sum is bounded by one and when $t \to +\infty$ it obviously converges to $\langle f \rangle'$ while the sums corresponding to other groups, when at least one of the indices belongs to I^c, will converge to zero because of the factor e^{-2t} in (10). By dominated convergence theorem we get convergence of expectations. $\quad\square$

References

[1] M. Aizenman, P. Contucci, On the stability of the quenched state in mean-field spin-glass models, J. Stat. Phys. 92 (5–6) (1998) 765–783.
[2] L.-P. Arguin, M. Aizenman, On the structure of quasi-stationary competing particles systems, Ann. Probab. 37 (3) (2009) 1080–1113.
[3] S. Ghirlanda, F. Guerra, General properties of overlap probability distributions in disordered spin systems. Towards Parisi ultrametricity, J. Phys. A 31 (46) (1998) 9149–9155.
[4] D. Panchenko, A connection between Ghirlanda–Guerra identities and ultrametricity, Ann. Probab. 38 (1) (2010) 327–347.
[5] D. Panchenko, On the Dovbysh–Sudakov representation result, Electron. Commun. Probab. 15 (2010) 330–338.
[6] M. Talagrand, Construction of pure states in mean-field models for spin glasses, Probab. Theory Related Fields 148 (3–4) (2010) 601–643.
[7] M. Talagrand, Spin Glasses: A Challenge for Mathematicians: Cavity and Mean Field Models, Springer-Verlag, Berlin, 2003.
[8] M. Talagrand, Mean-Field Models for Spin Glasses, second edition of [7], in press. First volume was published: http://www.springer.com/mathematics/probability/book/978-3-642-15201-6. Vol. 2 will appear in Ergebnisse Series, vol. 55.