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Number Theory/Functional Analysis

Absolutely continuous restrictions of a Dirac measure and non-trivial zeros of the Riemann zeta function

Restrictions absolument continues d'une mesure de Dirac et zéros non triviaux de la fonction zêta de Riemann

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ABSTRACT

It is shown that the Dirac measure $\delta(f) = f(1)$ defined on the Banach space $C([0,1])$ of complex valued continuous functions defined on the interval $[0,1]$, has an absolutely continuous restriction to an infinite dimensional subspace R of $C([0,1])$, that is

$$f(1) = \int_0^1 l(x) f(x)\,\mathrm{d}x, \quad \forall f \in R.$$

Each non-trivial zero of the Riemann zeta function determines a different Radon–Nikodym density $l \in L^1([0,1])$. The Riemann Hypothesis holds if and only if none of these densities belongs to $L^2([0,1])$ or if and only if R is dense in $L^2([0,1])$.

RÉSUMÉ

Nous montrons que la mesure de Dirac $\delta(f) = f(1)$ définie sur l'espace de Banach $C([0,1])$ de fonctions continues à valeurs complexes définies sur l'intervalle $[0,1]$, possède une restriction absolument continue sur un sous-espace de dimension infinie R de $C([0,1])$, c'est-à-dire

$$f(1) = \int_0^1 l(x) f(x)\,\mathrm{d}x, \quad \forall f \in R.$$

Chaque zéro non trivial de la fonction zêta de Riemann détermine une densité de Radon–Nikodym différente $l \in L^1([0,1])$. L'hypothèse de Riemann est vérifiée si et seulement si aucune de ces densités appartient à $L^2([0,1])$, ou si et seulement si R est dense dans l'espace $L^2([0,1])$.

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One of the first infinite dimensional Banach spaces to be studied was $C([0,1])$ equipped with the norm

$$\|f\| = \max_{x\in[0,1]} \big|F(x)\big|.$$

F. Riesz determined the dual $C([0,1])'$ of $C([0,1])$, proving that if $\varphi \in C([0,1])'$ then $\varphi(f) = \int_0^1 f(x)\,\mathrm{d}g(x)$, where $g : [0,1] \to \mathbb{C}$ is a function of bounded variation, that is $g = g_1 - g_2 + i(g_3 - g_4)$, where g_i, $1 \leqslant i \leqslant 4$, are non-decreasing and the integral is in the sense of Riemann–Stieltjes [6]. It is known that if $g : [0,1] \to \mathbb{R}$ is non-decreasing then there is a unique decomposition $g = g_s + g_d + g_{ac}$, where each term of the right-hand side is non-decreasing, g_s is continuous and singular in the sense that $g_s'(x) = 0$ a.e., g_d is constant except for jump discontinuities and g_{ac} is absolutely continuous [6]. One can recover g_s and g_d from their distributional derivatives but not from their ordinary derivatives. For g_{ac} the ordinary and distributional derivatives coincide. The terms g_d and g_{ac} admit a physical and probabilistic interpretation. Until now no use has been found for the term g_s. For the rest of this note we need the following reformulation of the Riemann Hypothesis (R.H.) [1,2], expressed in terms of the integral Hilbert–Schmidt, non-nuclear, non-normal operator defined on $L^2([0,1])$ by

$$[Af](\theta) = \int_0^1 \rho\left(\frac{\theta}{x}\right) f(x)\,\mathrm{d}x$$

(A also makes sense on $L^1([0,1])$) where ρ is the fractional part function given by $\rho(x) = x - [x]$, $x \in \mathbb{R}$, $[x] \in \mathbb{Z}$, $[x] \leqslant x < [x] + 1$, and whose Hilbert space adjoint A^*, by Fubini's theorem, takes the form

$$\big[A^* g\big](x) = \int_0^1 \rho\left(\frac{\theta}{x}\right) g(\theta)\,\mathrm{d}\theta.$$

Theorem 1. *R.H. holds if and only if* $\ker A = \{0\}$ *or if and only if* $h \notin \mathrm{R}(A)$, *where* $h(x) = x$.

By duality the condition $\ker A = \{0\}$ is equivalent to the statement that $\mathrm{R} = \mathrm{R}(A^*)$ is dense in $L^2([0,1])$.

Several properties of the operator A are studied in [1,2]. Our main claim, as stated in the abstract, will be proven showing that $\mathrm{R} \subset C([0,1])$, where A^* is the Hilbert space adjoint of A, and that

$$f(1) = \int_0^1 f(x) l(x)\,\mathrm{d}x, \quad \forall f \in \mathrm{R} \tag{1}$$

where $l \in L^1([0,1])$ obeys the equation $Al = h$.

First we note that if $s = \sigma + it$, $\sigma > -1$, $t \in \mathbb{R}$, then [1,4]

$$Ah^s = \frac{h}{s} - \frac{\zeta(s+1)}{s+1} h^{s+1} \tag{2}$$

and therefore $A(sh^s) = h$ if $\zeta(s+1) = 0$. If moreover $\zeta'(s+1) = 0$ then $A(-s^2 h^s \log h) = h$.

By the theorem, R.H. holds if and only if there is not $l \in L^2([0,1])$ such that $Al = h$; also $\ker A = \{0\}$ if and only if R is dense in $L^2([0,1])$. By (2) $\ker A^* = \{0\}$, since h is a cyclic vector for A by the Müntz theorem [5], and therefore R is infinite dimensional. Before proving that $\mathrm{R} \subset C([0,1])$ we show that (1) is a simple consequence of Fubini's theorem. Let us assume that $Al = h$, where $l \in L^1([0,1])$ and $f = A^*\varphi$, where $\varphi \in L^2([0,1])$. Then by Fubini's theorem we have

$$\int_0^1\int_0^1 \varphi(\theta)\rho\left(\frac{\theta}{x}\right) l(x)\,\mathrm{d}x\,\mathrm{d}\theta = \int_0^1 \varphi(\theta)\theta\,\mathrm{d}\theta = f(1) = \int_0^1 f(x) l(x)\,\mathrm{d}x. \tag{3}$$

We show next that $\mathrm{R} \subset C([0,1])$. From the formula

$$\rho\left(\frac{\theta}{x}\right) = \frac{\theta}{x} - \sum_{n=1}^{\infty} n\chi_{]\frac{\theta}{n+1},\frac{\theta}{n}]}(x), \quad x \in\,]0,1],\ \theta \in [0,1],$$

where χ_C is the characteristic function of the set C, one gets [1]

$$\big[A^*\varphi\big](x) = \frac{1}{x}\int_0^1 \theta\varphi(\theta)\,\mathrm{d}\theta - \sum_{k=1}^{\infty}\left\{\left[\sum_{n=1}^{k}\int_{nx}^{1}\varphi(\theta)\,\mathrm{d}\theta\right]\chi_{]\frac{1}{k+1},\frac{1}{k}]}(x)\right\}.$$

Therefore $A^*\varphi$ is continuous when restricted to $]0,1]$. The continuity of $A^*\varphi$ at 0 follows from a theorem of Fejér [7, vol. I, p. 49, T. 4.15] which implies that

$$\lim_{x\to 0^+}\int_0^1 \rho\left(\frac{\theta}{x}\right)\varphi(\theta)\,\mathrm{d}\theta = \frac{1}{2}\int_0^1 \varphi(\theta)\,\mathrm{d}\theta.$$

Therefore $A^*\varphi \in C([0,1])$ $\forall\varphi \in L^2([0,1])$ (the same result holds for $\varphi \in L^1([0,1])$). In concrete terms, we have proven the following theorem:

Theorem 2. *There exists an infinite dimensional subspace* R *of* $C([0,1])$, *such that for each non-trivial zero* $s+1$ *of the Riemann zeta function, there holds*

$$f(1) = \int_0^1 s x^s f(x)\,\mathrm{d}x, \quad \forall f \in \mathrm{R}.$$

Moreover, R *is dense in* $L^2([0,1])$ *if and only if* $R.H.$ *holds.*

It is not difficult to show that if $l \in L^1([0,1])$ is such that

$$\int_0^1 \psi(x) l(x)\,\mathrm{d}x = \psi(1), \quad \forall\psi \in \mathrm{R}\big(A^*\big)$$

then $A_\rho l = h$. But there are $\psi \in C([0,1]) \setminus R(A^*)$ for which the last equation holds true, for instance we can take $\psi = A^* f$ where $f \in L^1([0,1]) \setminus L^2([0,1])$. One can give explicitly a set of elements in $\mathrm{R}(A^*)$ that generate a dense subspace of $\mathrm{R}(A^*)$; if $0 \leqslant \alpha < \beta \leqslant 1$, then using the Fourier series for the Bernoulli polynomials $B_1(x)$ and $B_2(x)$ [3, T. 12.19] we get

$$\int_0^1 \rho\left(\frac{\theta}{x}\right)\chi_{[\alpha,\beta]}(\theta)\,\mathrm{d}\theta = \frac{1}{2}(\beta-\alpha) + \frac{x}{2}\left\{\rho\left(\frac{\beta}{x}\right) - \rho\left(\frac{\alpha}{x}\right)\right\}\left\{\rho\left(\frac{\beta}{x}\right) + \rho\left(\frac{\alpha}{x}\right) - 1\right\}.$$

To show that $\overline{\mathrm{R}(A^*)} = L^2([0,1])$ it is enough to prove that the characteristic function of a single interval is in $\overline{\mathrm{R}(A^*)}$ [4]. Finally, using the polynomials of Bernstein and an explicit formula given in [1,2] for $(\lambda + A)^{-1}h$ one can show that

$$\lim_{\lambda\to 0^+}\big\langle f, (\lambda + A)^{-1}h\big\rangle = f(1), \quad \forall f \in C\big([0,1]\big). \tag{4}$$

Using (4) and an expansion in Legendre polynomials for $(\lambda + A)^{-1}h$ it can be proven that

$$\lim_{\lambda\to 0^+}\big\|(\lambda + A)^{-1}h\big\| = \infty.$$

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