C. R. Acad. Sci. Paris, Ser. I 349 (2011) 437–439

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Algebraic Geometry/Analytic Geometry

Complex structures on products of circle bundles over complex manifolds

Structures complexes sur les produits de fibrés en cercles au dessus des variétés complexes

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ARTICLE INFO

Article history:
Received 1 December 2010
Accepted 17 February 2011
Available online 2 March 2011

Presented by Jean-Pierre Demailly

ABSTRACT

We propose, in this Note, a construction of complex structures on the product of two circle bundles associated to negative ample line bundles over flag varieties $X_i := G_i/P_i$, $i = 1, 2$, where the G_i are complex semisimple linear Lie groups and the $P_i \subset G_i$ are parabolic subgroups. The resulting manifold S is non-symplectic and hence non-Kählerian. We show that the group $\mathrm{Pic}^0(S)$ of topologically trivial holomorphic line bundles on S is isomorphic to $\mathbb{C}$.

RÉSUMÉ

Dans ce Note, on propose une construction de structures complexes sur le produit de deux fibré en cercles associés aux fibrés en droites, amples, négatifs sur des variétés drapeaux $X_i = G_i/P_i$, $i = 1, 2$, où les G_i sont des groupes de Lie linéaires connexes, complexes, semi-simples et les $P_i \subset G_i$ sont des sous-groupes paraboliques. La variété construite S n'est pas symplectique et donc n'est pas kählérienne. On démontre que le groupe $\mathrm{Pic}^0(S)$ des fibrés en droites holomorphes topologiquement triviaux est isomorphe aux nombres complexes $\mathbb{C}$.

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1. Introduction

Let $p_i : L_i \to X_i$ be a principal holomorphic $\mathbb{C}^*$-bundle over a compact complex manifold X_i, $i = 1, 2$. We shall denote by $\bar{L}_i$ the associated holomorphic line bundle over X_i and identify X_i with the image of the zero-cross section in $\bar{L}_i$ so that $L_i = \bar{L}_i \setminus X_i$.

Put a Hermitian metric on $\bar{L}_i$ and let $S(L_i) \subset L_i$ denote the space of norm 1 elements. Thus $S(L_i)$ is the total space of the principal $\mathbb{S}^1$-bundle over X_i determined by $\bar{L}_i$. We denote by L the $\mathbb{C}^* \times \mathbb{C}^*$-bundle $L_1 \times L_2 \to X_1 \times X_2$ and by $S(L)$ the principal $\mathbb{S}^1 \times \mathbb{S}^1$-bundle $S(L_1) \times S(L_2)$ over $X := X_1 \times X_2$. One obtains a complex structure on $S(L)$ as follows: Choose a complex number $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$. One has a proper holomorphic embedding $\mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*$ defined as $z \mapsto (\exp(2\pi\sqrt{-1}z), \exp(2\pi\sqrt{-1}\tau z))$. The restricted $\mathbb{C}$-action via this embedding on L is such that the quotient $L/\mathbb{C}$ is a complex manifold diffeomorphic to $S(L)$. The quotient map $L \to L/\mathbb{C}$ is the projection of a principal $\mathbb{C}$-bundle. Furthermore, with the induced complex structure on $S(L) \cong L/\mathbb{C}$, the differentiable bundle $S(L) \to X$ is the projection of a principal elliptic curve bundle over X with fiber and structure group $\mathbb{C}^* \times \mathbb{C}^*/\mathbb{C} =: E_\tau$ the elliptic curve with periods $\{1, \tau\}$. We refer

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.02.016

to the resulting complex structure as *scalar type*. Taking $X_i = \mathbb{P}^{n_i}$, $i = 1, 2$ and $\bar{L}_i$ to be tautological line bundles, one obtains the Calabi–Eckmann manifolds [2].

Now suppose that $\bar{L}_i \to X_i$ is a $T_i = (\mathbb{C}^*)^{n_i}$-equivariant bundle where the T_i-actions on L_i and X_i are holomorphic. Set $N = n_1 + n_2$ and $T = T_1 \times T_2 = (\mathbb{C}^*)^N$. We shall denote by $\epsilon_j : \mathbb{C}^* \subset (\mathbb{C}^*)^N$ the inclusion of the j-th factor and write $t\epsilon_j$ to denote $\epsilon_j(t)$ for $1 \leqslant j \leqslant N$. Thus any $t = (t_1, \ldots, t_N) \in T$ equals $\prod_{1 \leqslant j \leqslant N} t_j \epsilon_j$, and, under the exponential map $\mathbb{C}^N \to (\mathbb{C}^*)^N$, $\sum_{1 \leqslant j \leqslant N} z_j e_j$ maps to $\prod \exp(z_j)\epsilon_j$. (Here e_j denotes the standard basis vector of $\mathbb{C}^N$.)

We put a Hermitian metric on $\bar{L}_i$ which is invariant under action of the maximal compact subgroup $K_i = (\mathbb{S}^1)^{n_i} \subset T_i$.

Definition 1. Let d be a positive integer. We say that the $T_1 = (\mathbb{C}^*)^{n_1}$-action on L_1 is *d-standard* (or more briefly *standard*) if the following conditions hold:

(i) the restricted action of the diagonal subgroup $\Delta \subset T_1$ on L_1 is via the d-fold covering projection $\Delta \to \mathbb{C}^*$ onto the structure group $\mathbb{C}^*$ of $L_1 \to X_1$. (Thus if $d = 1$, the action of Δ coincides with that of the structure group of L_1.)
(ii) For any $0 \neq v \in L_1$ and $1 \leqslant j \leqslant n_1$ let $\nu_{v,j} : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as $t \mapsto \|t\epsilon_j.v\|$. Then $\nu'_{v,j}(t) > 0$ for all t unless $\mathbb{R}_+\epsilon_j$ is contained in the isotropy at v.

Let $\lambda = (\lambda_1, \ldots, \lambda_N) \in Lie(T) \cong \mathbb{C}^N$. We recall the definition of weak hyperbolicity [4]. Let $\lambda = (\lambda_1, \ldots, \lambda_N)$, $N = n_1 + n_2$. One says that λ satisfies the *weak hyperbolicity condition of type* (n_1, n_2) if

$$0 \leqslant \arg(\lambda_i) < \arg(\lambda_j) < \pi, \quad 1 \leqslant i \leqslant n_1 < j \leqslant N. \tag{1}$$

If $\lambda_j = \lambda_1\ \forall j \leqslant n_1$, $\lambda_j = \lambda_N\ \forall j > n_1$, with $0 \leqslant \arg(\lambda_1) < \arg(\lambda_N) < \pi$, we say that λ is of scalar type.

Definition 2. Suppose that the $T_i = (\mathbb{C}^*)^{n_i}$-action on L_i is d_i-standard for some $d_i \geqslant 1$, $i = 1, 2$, and let $\lambda \in \mathbb{C}^N = Lie(T)$. The analytic homomorphism $\alpha_\lambda : \mathbb{C} \to T = (\mathbb{C}^*)^N$ defined as $\alpha_\lambda(z) = \exp(z\lambda)$ is said to be *admissible* if λ satisfies the weak hyperbolicity condition (1) of type (n_1, n_2) above. We denote by the image of α_λ by $\mathbb{C}_\lambda$. If α_λ is admissible, we say that the $\mathbb{C}_\lambda$-action on $L = L_1 \times L_2$ is of *diagonal type*.

It is not difficult to see that the restricted $\mathbb{C}_\lambda$-action on L is free when it is admissible. We have the following result:

Theorem 3. *With the above notations, suppose that $\alpha_\lambda : \mathbb{C} \to T$ defines an admissible action of $\mathbb{C}$ of diagonal type on L. Then $L/\mathbb{C}$ is a (Hausdorff) complex analytic manifold and the quotient map $L \to L/\mathbb{C}$ is the projection of a holomorphic principal $\mathbb{C}$-bundle. Furthermore, each $\mathbb{C}$-orbit meets $S(L)$ transversely at a unique point so that $L/\mathbb{C}$ is diffeomorphic to $S(L)$.*

We denote by $S_\lambda(L)$ the manifold $S(L)$ with the complex structure induced from $L/\mathbb{C}_\lambda$. The complex structure on $S_\lambda(L)$ will be said to be of *diagonal type*.

The proof of Theorem 3 follows mainly the ideas of Loeb and Nicolau [4].

2. The main results

For $i = 1, 2$, let G_i be a simply-connected complex semisimple linear algebraic group. Fix a maximal torus T_i, a Borel subgroup $B_i \supset T_i$ and a parabolic $P_i \supset B_i$. We assume that $P_i \neq G_i$. Let $\bar{L}_i \to X_i$ be any negative ample line bundle over the flag variety $X_i := G_i/P_i$. The T_i-action on L_i is not standard. However, we let $\tilde{G}_i := G_i \times \mathbb{C}^*$, $\tilde{T}_i := T_i \times \mathbb{C}^*$, and so on, where the second factor, $\mathbb{C}^*$, is just the structure group of $\bar{L}_i$. Then $\bar{L}_i$ is a $\tilde{G}_i$-equivariant bundle over $X_i = \tilde{G}_i/\tilde{P}_i$. We equip $\bar{L}_i$ with a Hermitian metric invariant under the action of a maximal compact subgroup of $\tilde{G}_i$ which contains the maximal compact subgroup of $\tilde{T}_i$. It can be shown that there exists an isomorphism $\tilde{T}_i \cong (\mathbb{C}^*)^{n_i}$ and a positive integer d_i such that the action of $\tilde{T}_i$ on L_i is d_i-standard.

Let $\tilde{G} = \tilde{G}_1 \times \tilde{G}_2$, etc. Let $\tilde{B}_u$ denote the unipotent radical of $\tilde{B}$ so that $\tilde{B} = \tilde{T}.\tilde{B}_u = (\tilde{T}_1 \times \tilde{T}_2).\tilde{B}_u$. Let $\lambda \in Lie(\tilde{B})$ and let $\lambda = \lambda_s + \lambda_u$ be its Jordan decomposition where $\lambda_s \in Lie(\tilde{T}) \cong \mathbb{C}^N$, $\lambda_u \in Lie(\tilde{B}_u)$ where $N = n_1 + n_2$. We have an analytic homomorphism $\alpha_\lambda : \mathbb{C} \to \tilde{B}$ defined as $z \mapsto \exp(z\lambda) = \exp(z\lambda_s)\exp(z\lambda_u)$. This defines an action of $\mathbb{C}$ on $L = L_1 \times L_2$. We say that this action is *admissible* if $\lambda_s \in \mathbb{C}^N$ satisfies the weak hyperbolicity condition of type (n_1, n_2). Such an admissible $\mathbb{C}$ action is free and is said to be of *linear type*. We have the following theorem:

Theorem 4. *We keep the above notations. The orbit space $L/\mathbb{C}$ of an admissible linear type $\mathbb{C}$-action on $L = L_1 \times L_2$ defined by $\lambda \in Lie(\tilde{B})$ is a Hausdorff complex manifold and the quotient map $L \to L/\mathbb{C}$ is the projection of a principal $\mathbb{C}$-bundle. Furthermore, $L/\mathbb{C}$ is diffeomorphic to $S(L) = S(L_1) \times S(L_2)$.*

The resulting complex structure on $S(L)$ is said to be of *linear type* and $S_\lambda(L)$ will denote $S(L)$ with the induced complex structure.

Using the fact that the $(G_i/P_i; \bar{L}_i^\vee)$ are arithmetically Cohen–Macaulay [6] and projectively normal [5], we show that $H^q(L_i; \mathcal{O}_{L_i})$ is Hausdorff and that it vanishes for $1 \leqslant q < \dim X_i$ (see [1, Chapter II]). We use the Künneth formula due to A. Cassa [3] to obtain a vanishing theorem for the cohomology groups $H^q(L; \mathcal{O}_L)$. Using this result we obtain the following

Theorem 5. *Let $S_\lambda(L)$ be as in Theorem 4. Then $H^q(S_\lambda(L); \mathcal{O}) = 0$ provided $q \notin \{0, 1, \dim X_i, \dim X_i + 1, \dim X_1 + \dim X_2, \dim X_1 + \dim X_2 + 1; i = 1, 2\}$.*

The following result describes the Picard group of $S_\lambda(L)$:

Theorem 6. *Let $X_i = G_i/P_i$ where P_i is any parabolic subgroup and let $\bar{L}_i \to X_i$ be a negative ample line bundle, $i = 1, 2$. We assume that, when $X_i = \mathbb{P}^1$, the bundle $\bar{L}_i$ is a generator of $\mathrm{Pic}(X_i)$. Then $\mathrm{Pic}^0(S_\lambda(L)) \cong \mathbb{C}$. If the P_i are maximal parabolics and the $\bar{L}_i$ are generators of $\mathrm{Pic}(X_i) \cong \mathbb{Z}$, then $\mathrm{Pic}(S_\lambda(L)) \cong \mathrm{Pic}^0(S_\lambda(L)) \cong \mathbb{C}$.*

Finally we obtain the following:

Theorem 7. *Let L_i be the negative ample generator of $\mathrm{Pic}(G_i/P_i) \cong \mathbb{Z}$ where P_i is a maximal parabolic subgroup of G_i, $i = 1, 2$. Assume that $S_\lambda(L)$ is of diagonal type. Then the field $\kappa(S_\lambda(L))$ of meromorphic functions of $S_\lambda(L)$ is purely transcendental over $\mathbb{C}$. The transcendence degree of $\kappa(S_\lambda(L))$ is less than $\dim S_\lambda(L)$.*

Theorem 6 is a generalization of a result of Loeb and Nicolau who considered complex structures on products of odd-dimensional spheres. However, they considered complex structures which are more general than linear types.

Suppose that $X_i = G_i/P_i$ where P_i are maximal parabolic and $\bar{L}_i$ are negative ample generators of $\mathrm{Pic}(X_i) = \mathbb{Z}$. Then, it is a well-known result due to Wang [8] that $S(L)$ admits a complex structure invariant under the action of a maximal compact subgroup of $G = G_1 \times G_2$. These Wang manifolds correspond to scalar type complex structures (cf. [7]).

The detailed proofs of results announced here will be published elsewhere.

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