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Phase transitions for XY-model on the Cayley tree of order three in quantum Markov chain scheme

Transitions de phases pour un modèle XY sur un arbre de Cayley d'ordre trois dans un schéma de chaines de Markov quantiques

Farrukh Mukhamedov, Mansoor Saburov

Department of Computational & Theoretical Sciences, Faculty of Science, International Islamic University Malaysia, PO Box 141, 25710 Kuantan, Pahang, Malaysia

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ABSTRACT

In the present Note we study forward Quantum Markov Chains (QMC) defined on a Cayley tree. Using the tree structure of graphs, we give a construction of quantum Markov chains on the Cayley tree. By means of such constructions we prove the existence of a phase transition for the XY-model on a Cayley tree of order three in QMC scheme. By the phase transition we mean the existence of two distinct QMC for the given family of interaction operators $\{K_{\langle x,y\rangle}\}$.

RÉSUMÉ

Dans cette Note on étudie des chaines de Markov directes (QMC) définies sur un arbre de Cayley. En utilisant la structure en arbre des graphes on donne une construction de chaines de Markov quantiques sur un arbre de Cayley. Au moyen de telles constructions on démontre l'existence d'une transition de phases pour un modèle XY sur un arbre de Cayley d'ordre trois dans un schéma QMC. La transition de phases correspond ici à l'existence de deux QMC distinctes pour une famille $\{K_{\langle x,y\rangle}\}$ d'opérateurs d'interactions.

1. Introduction

It is known that Markov fields play an important role in classical probability, in physics, in biological and neurological models and in an increasing number of technological problems such as image recognition. Therefore, it is quite natural to forecast that the quantum analogue of these models will also play a relevant role. The quantum analogues of Markov processes were first constructed in [1], where the notion of quantum Markov chain on infinite tensor product algebras was introduced.

A first attempt to construct a quantum analogue of classical Markov fields has been done in [10,2]. These papers extend to field the notion of *quantum Markov state* introduced in [3] as a sub-class of the *quantum Markov chains* introduced in [1]. Note that in the mentioned papers quantum Markov fields were considered over multidimensional integer lattice $\mathbb{Z}^d$. This lattice has so called amenability property. Therefore, it is natural to investigate quantum Markov fields over non-amenable lattices. One of the simplest non-amenable lattices is a Cayley tree. First attempts to investigate quantum Markov chains

E-mail addresses: far75m@yandex.ru, farrukh_m@iiu.edu.my (F. Mukhamedov), msaburov@gmail.com (M. Saburov).

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over such trees were done in [6], such studies were related to investigate thermodynamic limit of valence-bond-solid models on a Cayley tree [8]. There, it was constructed finitely correlated states as ground states of VBS model on Cayley tree. The mentioned considerations naturally suggest the study of the following problem: the extension to field the notion of quantum Markov chain.

If a tree is not one-dimensional lattice, then the existence of a phase transition for quantum Markov chains constructed over such a tree is expected (from physical point of view). Note that phase transitions in a quantum setting play an important role to understand quantum spin systems (see for example [9]). In the paper using a tree structure of graphs, we give a construction of quantum Markov chains on Cayley tree. By means of such constructions we prove the existence of a phase transition for the XY-model on a Cayley tree of order three in QMC scheme.

2. Preliminaries

Let $\Gamma_+^k=(L,E)$ be a semi-infinite Cayley tree of order $k\geqslant 1$ with the root x^0 (whose each vertex has exactly $k+1$ edges, except for the root x^0, which has k edges). Here L is the set of vertices and E is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l=\langle x,y\rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x,x_1\rangle,\dots,\langle x_{d-1},y\rangle$ is called a *path* from the point x to the point y. The distance $d(x,y), x,y\in V$, on the Cayley tree, is the length of the shortest path from x to y. Recall a coordinate structure in Γ_+^k: every vertex x (except for x^0) of Γ_+^k has coordinates $(i_1,\dots,i_n)$, here $i_m\in\{1,\dots,k\}$, $1\leqslant m\leqslant n$ and for the vertex x^0 we put (0). Namely, the symbol (0) constitutes level 0, and the sites $(i_1,\dots,i_n)$ form level n (i.e. $d(x^0,x)=n$) of the lattice. Let us set $W_n=\{x\in L:\ d(x,x_0)=n\}$, $\Lambda_n=\bigcup_{k=0}^n W_k$, $\Lambda_n^c=\bigcup_{k=n}^\infty W_k$, $E_n=\{\langle x,y\rangle\in E:\ x,y\in\Lambda_n\}$.

The algebra of observables $\mathcal{B}_x$ for any single site $x\in L$ will be taken as the algebra M_d of the complex $d\times d$ matrices. The algebra of observables localized in the finite volume $\Lambda\subset L$ is then given by $\mathcal{B}_\Lambda=\bigotimes_{x\in\Lambda}\mathcal{B}_x$. As usual if $\Lambda^1\subset\Lambda^2\subset L$, then $\mathcal{B}_{\Lambda^1}$ is identified as a subalgebra of $\mathcal{B}_{\Lambda^2}$ by tensoring with unit matrices on the sites $x\in\Lambda^2\setminus\Lambda^1$. Note that, in the sequel, by $\mathcal{B}_{\Lambda,+}$ we denote positive part of $\mathcal{B}_\Lambda$. The full algebra $\mathcal{B}_L$ of the tree is obtained in the usual manner by an inductive limit $\mathcal{B}_L=\overline{\bigcup_{\Lambda_n}\mathcal{B}_{\Lambda_n}}$. In what follows, by $\mathcal{S}(\mathcal{B}_\Lambda)$ we will denote the set of all states defined on the algebra $\mathcal{B}_\Lambda$.

Consider a triplet $\mathcal{C}\subset\mathcal{B}\subset\mathcal{A}$ of unital C^*-algebras. Recall that a *quasi-conditional expectation* with respect to the given triplet is a completely positive identity preserving linear map $\mathcal{E}:\mathcal{A}\to\mathcal{B}$ such that $\mathcal{E}(ca)=c\mathcal{E}(a)$, for all $a\in\mathcal{A}, c\in\mathcal{C}$.

A state φ on $\mathcal{B}_L$ is called a *forward quantum d-Markov chain* (*QMC*) (see [5]), associated to $\{\Lambda_n\}$, if for each Λ_n, there exist a quasi-conditional expectation $\mathcal{E}_{\Lambda_n^c}$ with respect to the triplet $\mathcal{B}_{\Lambda_{n+1}^c}\subseteq\mathcal{B}_{\Lambda_n^c}\subseteq\mathcal{B}_{\Lambda_{n-1}^c}$ and a state $\hat\varphi_{\Lambda_n^c}\in\mathcal{S}(\mathcal{B}_{\Lambda_n^c})$ such that for any $n\in\mathbb{N}$ one has $\hat\varphi_{\Lambda_n^c}|\mathcal{B}_{\Lambda_{n+1}\setminus\Lambda_n}=\hat\varphi_{\Lambda_{n+1}^c}\circ\mathcal{E}_{\Lambda_{n+1}^c}|\mathcal{B}_{\Lambda_{n+1}\setminus\Lambda_n}$ and $\varphi=\lim_{n\to\infty}\hat\varphi_{\Lambda_n^c}\circ\mathcal{E}_{\Lambda_n^c}\circ\mathcal{E}_{\Lambda_{n-1}^c}\circ\cdots\circ\mathcal{E}_{\Lambda_1^c}$ in the weak-* topology.

3. Constructions of quantum *d*-Markov chains on the Cayley tree

Let us rewrite the elements of W_n in the following order, i.e. $\overrightarrow{W}_n:=(x_{W_n}^{(1)},x_{W_n}^{(2)},\dots,x_{W_n}^{(|W_n|)})$. Note that $|W_n|=k^n$. Vertices $x_{W_n}^{(1)},x_{W_n}^{(2)},\dots,x_{W_n}^{(|W_n|)}$ of W_n can be represented in terms of the coordinate system as follows:

$$\begin{aligned}
&x_{W_n}^{(1)}=(1,1,\dots,1,1),\qquad x_{W_n}^{(2)}=(1,1,\dots,1,2),\qquad \dots,\qquad x_{W_n}^{(k)}=(1,1,\dots,1,k,),\\
&x_{W_n}^{(k+1)}=(1,1,\dots,2,1),\qquad x_{W_n}^{(2)}=(1,1,\dots,2,2),\qquad \dots,\qquad x_{W_n}^{(2k)}=(1,1,\dots,2,k),\qquad \dots,\\
&x_{W_n}^{(|W_n|-k+1)}=(k,k,\dots,k,1),\qquad x_{W_n}^{(|W_n|-k+2)}=(k,k,\dots,k,2),\qquad \dots,\qquad x_{W_n}^{|W_n|}=(k,k,\dots,k,k).
\end{aligned}$$

Analogously, for a given vertex x, we shall use the following notation for the set of direct successors of x:

$$\overrightarrow{S(x)}:=\big((x,1),(x,2),\dots,(x,k)\big),\qquad \overleftarrow{S(x)}:=\big((x,k),(x,k-1),\dots,(x,1)\big).$$

Assume that for each edge $\langle x,y\rangle\in E$ of the tree is assigned an operator $K_{\langle x,y\rangle}\in\mathcal{B}_{\{x,y\}}$. We would like to define a state on $\mathcal{B}_{\Lambda_n}$ with boundary conditions $w_0\in\mathcal{B}_{(0),+}$ and $\mathbf{h}=\{h_x\in\mathcal{B}_{x,+}\}_{x\in L}$.

Let us denote

$$K_{[m-1,m]}:=\prod_{x\in\overrightarrow{W}_{m-1}}\prod_{y\in\overrightarrow{S(x)}}K_{\langle x,y\rangle},\qquad \mathbf{h}_n^{1/2}:=\prod_{x\in\overrightarrow{W}_n}h_x^{1/2},\qquad \mathbf{h}_n:=\mathbf{h}_n^{1/2}\big(\mathbf{h}_n^{1/2}\big)^*,$$

$$K_n:=\omega_0^{1/2}K_{[0,1]}K_{[1,2]}\cdots K_{[n-1,n]}\mathbf{h}_n^{1/2},\qquad \mathcal{W}_{n]}:=K_nK_n^*.$$

It is clear that $\mathcal{W}_{n]}$ is positive.

In what follows, by $\mathrm{Tr}_\Lambda:\mathcal{B}_L\to\mathcal{B}_\Lambda$ we mean normalized partial trace, for any $\Lambda\subseteq_{\text{fin}}L$. For the sake of shortness we put $\mathrm{Tr}_{n]}:=\mathrm{Tr}_{\Lambda_n}$. Define a positive functional $\varphi_{w_0,\mathbf{h}}^{(n,f)}$ on $\mathcal{B}_{\Lambda_n}$ by $\varphi_{w_0,\mathbf{h}}^{(n,f)}(a)=\mathrm{Tr}(\mathcal{W}_{n+1]}(a\otimes\mathbb{1}_{W_{n+1}}))$, for every $a\in\mathcal{B}_{\Lambda_n}$, where $\mathbb{1}_{W_n+1}=\bigotimes_{y\in W_{n+1}}\mathbb{1}$. Note that here, Tr is a normalized trace on $\mathcal{B}_L$. To get an infinite-volume state $\varphi^{(f)}$ on $\mathcal{B}_L$ such that $\varphi^{(f)}\lceil_{\mathcal{B}_{\Lambda_n}}=\varphi_{w_0,\mathbf{h}}^{(n,f)}$, we need to impose some constrains to the boundary conditions $\{w_0,\mathbf{h}\}$ so that the functionals $\{\varphi_{w_0,\mathbf{h}}^{(n,f)}\}$ satisfy the compatibility condition, i.e. $\varphi_{w_0,\mathbf{h}}^{(n+1,f)}\lceil_{\mathcal{B}_{\Lambda_n}}=\varphi_{w_0,\mathbf{h}}^{(n,f)}$.

Theorem 3.1. *Let the boundary conditions* $w_0 \in \mathcal{B}_{(0),+}$ *and* $\mathbf{h} = \{h_x \in \mathcal{B}_{x,+}\}_{x\in L}$ *satisfy the following conditions*:

$$\mathrm{Tr}(w_0 h_0) = 1, \qquad \mathrm{Tr}_{x]}\left[\prod_{y\in \overrightarrow{S(x)}} K_{\langle x,y\rangle} \prod_{y\in \overrightarrow{S(x)}} h_y \prod_{y\in \overleftarrow{S(x)}} K^*_{\langle x,y\rangle}\right] = h_x \quad \textit{for every } x \in L. \tag{1}$$

Then the functionals $\{\varphi^{(n,f)}_{w_0,\mathbf{h}}\}$ *satisfy the compatibility condition. Moreover, there is a unique forward quantum d-Markov chain* $\varphi^{(b)}_{w_0,\mathbf{h}}$ *on* $\mathcal{B}_L$ *such that* $\varphi^{(f)}_{w_0,\mathbf{h}} = w - \lim_{n\to\infty} \varphi^{(n,f)}_{w_0,\mathbf{h}}$.

Our goal in this Note is to establish existence of phase transition for the given family $\{K_{\langle x,y\rangle}\}$ of operators. Heuristically, the phase transition means the existence of two distinct QMC for the given $\{K_{\langle x,y\rangle}\}$. Let us provide more exact definition.

We say that there exists *a phase transition* for a family of operators $\{K_{\langle x,y\rangle}\}$ if (1) have at least two $(w_0, \{h_x\}_{x\in L})$ and $(\bar{w}_0, \{\bar{h}_x\}_{x\in L})$ solutions such that the corresponding quantum d-Markov chains $\varphi_{w_0,\mathbf{h}}$ and $\varphi_{\bar{w}_0,\bar{\mathbf{h}}}$ are not quasi-equivalent. Otherwise, we say there is no phase transition.

4. Quantum *d*-Markov chains associated with *XY*-model

In this section, we define the model and shall formulate main results of the Note. In what follows we consider a semi-infinite Cayley tree $\Gamma^3_+ = (L, E)$ of order 3. Our starting C^*-algebra is the same $\mathcal{B}_L$ but with $\mathcal{B}_x = M_2(\mathbb{C})$ for $x \in L$. By $\sigma_x^{(u)}, \sigma_y^{(u)}, \sigma_z^{(u)}$ we denote the Pauli spin operators for at site $u \in L$.

For every edge $\langle u, v\rangle \in E$ put

$$K_{\langle u,v\rangle} = \exp\left\{\frac{\beta}{2}\left(\sigma_x^{(u)}\sigma_x^{(v)} + \sigma_y^{(u)}\sigma_y^{(v)}\right)\right\}, \quad \beta > 0. \tag{2}$$

Such kind of Hamiltonian is called *quantum XY-model* per edge $\langle x, y\rangle$.

The main results of the paper concern existence of the phase transition for the model (2). Namely, we have

Theorem 4.1. *Let* $\{K_{\langle x,y\rangle}\}$ *be given by* (2) *on the Cayley tree of order three. Then there are two positive numbers* β_* *and* β^* *such that*

(i) *if* $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$, *then there is a unique forward quantum d-Markov chain associated with* (2);
(ii) *if* $\beta \in (\beta_*, \beta^*)$, *then there is a phase transition for given model.*

Remark 1. If we consider the same model (2) on the Cayley tree of order two, then there is only a unique forward quantum d-Markov chain for all values of $\beta > 0$ (see [4]). Hence, the main result of the present paper totally differs from [4], and shows by increasing the dimension of the tree we are getting the phase transition.

To prove the theorem, we need first the existence of forward QMC for the model. To this end, we show the existence of the solutions of Eqs. (1) with properties $h_u = h_v$ for every $u, v \in W_n$. It turns out that the equations lead to study the following dynamical system $F : \Delta \ni (x, y) \to (x', y') \in \mathbb{R}^2_+$

$$B_2\left(x'\right)^3 + A_2 x'\left(y'\right)^2 = x, \qquad B_1\left(x'\right)^2 y' + A_1\left(y'\right)^3 = y, \tag{3}$$

where $\Delta = \{(x, y) \in \mathbb{R}^2_+ \colon x > y\}$ and $A_1 = \sinh^3\beta\cosh\beta$, $B_1 = \sinh\beta\cosh^2\beta(1 + \cosh\beta + \cosh^2\beta)$, $A_2 = \sinh^2\beta\cosh^2\beta(1 + 2\cosh\beta)$, $B_2 = \cosh^6\beta$.

Note that elements of the set Δ guarantee positivity of the corresponding boundary conditions $\mathbf{h} = \{h_z \in \mathcal{B}_{z,+}\}_{z\in L}$. Therefore, we will study the dynamical system (3) in the domain Δ.

Using that the function defined by $g_\beta(t) = (A_1 t^3 + B_1 t)/(A_2 t^2 + B_2)$ is increasing on $[0, 1]$, we may prove that the system of Eqs. (3) has at most one solution with respect to $(x', y') \in \Delta$, whenever $x, y \in \Delta$. Therefore, the system of Eqs. (3) gives an implicit form of variables x', y'.

Remark 2. It is worth noting that the dynamical system $F : \Delta \to \Delta$ defined by (3) is well-defined if and only if $\frac{y}{x} \in g_\beta([0, 1])$, and moreover it can be written as follows:

$$x_1' = \sqrt[3]{\frac{x}{B_2 + A_2(g_\beta^{-1}(\frac{y}{x}))^2}}, \qquad y_1' = \sqrt[3]{\frac{y(g_\beta^{-1}(\frac{y}{x}))^2}{B_1 + A_1(g_\beta^{-1}(\frac{y}{x}))^2}}. \tag{4}$$

Examining the dynamical system (3) we derive that there exist two positive numbers β_* and β^* such that if $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$ then there is a unique fixed point $(\alpha_0, 0)$ of (3) in Δ, where $\alpha_0 = 1/\cosh^3\beta$. If $\beta \in (\beta_*, \beta^*)$ then there are two fixed points in the domain Δ, which are $(\alpha_0, 0)$ and $\gamma := (\mathbf{u}, \mathbf{v})$.

Investigating the dynamical system (4) we prove that if $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$ then Eqs. (1) have only the following parametrical solutions $(w_0(\alpha), \{h_x(\alpha)\})$ given by $w_0(\alpha) = \frac{1}{\alpha}\mathbb{1}$, $h_x^{(n)}(\alpha) = \alpha_0 \sqrt[3^n]{\alpha \cosh^3 \beta}\mathbb{1}$ for every $x \in V$, here α is any positive real number. Note that the boundary conditions corresponding to the fixed point of (3) are the following ones: $w_0(\alpha) = \frac{1}{\alpha_0}\mathbb{1}$, $h_x^{(n)}(\alpha_0) = \alpha_0\mathbb{1}$. Let us consider the state $\varphi^{(n,b)}_{w_0(\alpha),\mathbf{h}(\alpha)}$ corresponding to the solution $(w_0(\alpha), \{h_x^{(n)}(\alpha)\})$. By definition we have

$$\varphi^{(n,f)}_{w_0(\alpha),\mathbf{h}(\alpha)}(x) = \frac{\left(\sqrt[3^{n+1}]{\alpha \cosh^3 \beta}\right)^{3^{n+1}}}{\alpha (\cosh^3 \beta)^{3^{n+1}}} \operatorname{Tr}\left(\prod_{i=0}^{n-1} K_{[i,i+1]} \prod_{i=0}^{n-1} K^*_{[n-1-i,n-i]} x\right) = \varphi^{(n,f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}(x),$$

for any α. Hence, from the definition of quantum d-Markov chain we find that $\varphi^{(f)}_{w_0(\alpha),\mathbf{h}(\alpha)} = \varphi^{(f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}$, which yields that the uniqueness of forward quantum d-Markov chain associated with the model (2).

Now let us for the sake of simplicity denote $\sigma_0 := \mathbb{1}$, $\sigma_1 := \sigma_x$, $\sigma_2 := \sigma_y$, $\sigma_3 := \sigma_z$.

Assume that $\beta \in (\beta_*, \beta^*)$, then the dynamical system (3) has two fixed points. Hence, the corresponding solutions of Eqs. (1) can be written as follows: $(w_0(\alpha_0), \{h_x(\alpha_0)\})$ and $(w_0(\gamma), \{h_x(\gamma)\})$, where

$$w_0(\alpha_0) = \frac{1}{\alpha_0}\sigma_0, \qquad h_x(\alpha_0) = \alpha_0 \sigma_0^{(x)}, \qquad w_0(\gamma) = \frac{1}{\mathbf{u}}\sigma_0, \qquad h_x(\gamma) = \mathbf{u}\sigma_0^{(x)} + \mathbf{v}\sigma_1^{(x)}.$$

By $\varphi^{(f)}_{w_0(\alpha_0),\mathbf{h}(\gamma)}$, $\varphi^{(f)}_{w_0(\gamma),\mathbf{h}(\gamma)}$ we denote the corresponding forward quantum d-Markov chains. To prove the existence of the phase transition, we need to show that these two states are not quasi-equivalent.

Let us consider the following elements:

$$\sigma_1^{\overrightarrow{S(x)},1} := \sigma_1^{(1)} \otimes \sigma_0^{(2)} \otimes \sigma_0^{(3)}, \qquad a_{\sigma_1}^{\Lambda_{n+1}} := \bigotimes_{i=0}^{n}\left(\bigotimes_{x \in \overrightarrow{W}_i} \sigma_0^{(x)}\right) \otimes \sigma_1^{\overrightarrow{S(x^{(1)}_{W_n})},1} \otimes \left(\bigotimes_{x \in \overrightarrow{W_{n+1}} \setminus \overrightarrow{S(x^{(1)}_{W_n})}} \sigma_0^{(x)}\right).$$

Then we prove that

$$\varphi^{(f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}\left(a_{\sigma_1}^{\Lambda_{N+1}}\right) = 0, \qquad \varphi^{(f)}_{w_0(\gamma),\mathbf{h}(\gamma)}\left(a_{\sigma_1}^{\Lambda_{N+1}}\right) \to \frac{x_1^2 h_1 + x_1 y_1 h_2 \sinh \beta}{\mathbf{u}(x_1^2 + y_1^2 \sinh \beta)} \quad \text{as } N \to \infty, \tag{5}$$

where $N \in \mathbb{N}$ and x_1, y_1, h_0, h_1 are some positive numbers, which depend on $\mathbf{u}$, $\mathbf{v}$ and β. Hence, by means of (5) with Corollary 2.6.11 [7] we obtain that the states $\varphi^{(f)}_{w_0,\mathbf{h}(\alpha_0)}$ and $\varphi^{(f)}_{w_0,\mathbf{h}(\gamma)}$ are not quasi-equivalent. Hence, this completes the proof of Theorem 4.1.

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