C. R. Acad. Sci. Paris, Ser. I 349 (2011) 297–301

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Harmonic Analysis

Change of angle in tent spaces

Changement d'angle dans les espaces de tentes

Pascal Auscher

Univ. Paris-Sud and CNRS, laboratoire de mathématiques, UMR 8628, Orsay 91405 cedex, France

ARTICLE INFO

Article history:
Received 19 January 2011
Accepted 26 January 2011

Presented by Yves Meyer

ABSTRACT

We prove sharp bounds for the equivalence of norms in tent spaces with respect to changes of angles. Some applications are given.

RÉSUMÉ

On propose de démontrer les comparaisons précises entre les normes dans le même espace de tentes avec deux angles différents. On donne quelques conséquences.

Version française abrégée

Si $0 < p < \infty$ et $\alpha > 0$, l'espace des tentes $T^{p,2}_\alpha$ est défini dans [4] comme l'espace des fonctions g localement de carré integrable dans le demi-espace supérieur ouvert $\mathbb{R}^{n+1}_+$ telles que $A^{(\alpha)}g \in L^p$ où $A^{(\alpha)}g$ est définie par (1). Si $p = \infty$, on définit $T^{\infty,2}_\alpha$ par la condition de Carleson (3) sur les tentes $T_\alpha B$ et la norme est la meilleure constante C dans cette inégalité. Le paramètre α se décrit comme le demi-angle des cônes d'appui dans la définition de la fonctionnelle d'aire $A^{(\alpha)}$. Le choix des normalisations en fonction de α obéit à une cohérence d'échelle pour tous les p.

Un des points de départ de la théorie de [4] est que pour p fixé l'espace ne dépend pas de $\alpha > 0$ et que les (quasi-)normes $\|A^{(\alpha)}g\|_p$ sont deux à deux équivalentes. Ce résultat est utilisé alors pour obtenir les décompositions atomiques ($p \leqslant 1$), la dualité ($1 \leqslant p < \infty$) et l'interpolation complexe et réelle ($0 < p \leqslant \infty$).

Il est récemment apparu utile de connaître la dépendance précise en terme des angles dans l'équivalence des normes. Les espaces de tentes sont en effet intensivement utilisés dans la nouvelle théorie des espaces de Hardy associés aux opérateurs vérifiant des estimations de Gaffney–Davies découlant des travaux [2,7]. Cette dépendance se retrouve aussi cruciale pour étudier l'opérateur de régularité maximale dans les espaces de tentes et ses applications aux EDP paraboliques [3].

L'argument de [4], provenant de [6], ne donne pas la dépendance optimale. Dans [8], il est démontré $\|g\|_{T^{p,2}_\alpha} \leqslant C(1 + \log\alpha)\alpha^{n/\tau}\|g\|_{T^{p,2}_1}$ pour $1 < p < \infty$ et $\alpha \geqslant 1$, où $\tau = \min(p, 2)$ et C dépend de n et p, dans un cadre à valeurs UMD, ce qui explique les restrictions sur p. La démonstration n'est pas élémentaire. Il est aussi démontré que le croissance polynomiale $\alpha^{n/\tau}$ est la meilleure possible dans le cas qui nous occupe ici.

Nous obtenons pour $0 < p \leqslant \infty$ et $\alpha, \beta > 0$

$$C\underline{h}(p, \alpha/\beta)\|g\|_{T^{p,2}_\alpha} \leqslant \|g\|_{T^{p,2}_\beta} \leqslant C'\bar{h}(p, \alpha/\beta)\|g\|_{T^{p,2}_\alpha}$$

E-mail address: pascal.auscher@math.u-psud.fr.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.01.023

où C, C' dépendent de n, p, $\underline{h}(p,\alpha) = \min(\alpha^{-n/2}, \alpha^{-n/p})$, $\bar{h}(p,\alpha) = \max(\alpha^{-n/2}, \alpha^{-n/p})$. Les exposants sont les meilleurs possibles. En particulier si $\alpha > 1$, $\tau = \min(2, p)$ et $\sigma = \max(2, p)$, $\|g\|_{T^{p,2}_\alpha} \leqslant C\alpha^{n/\tau}\|g\|_{T^{p,2}_1}$ et $\|g\|_{T^{p,2}_1} \leqslant C\alpha^{-n/\sigma}\|g\|_{T^{p,2}_\alpha}$. Remarquer que la deuxième inégalité améliore $\|g\|_{T^{p,2}_1} \leqslant \|g\|_{T^{p,2}_\alpha}$ découlant simplement de la géométrie des cônes d'appui.

La preuve de ces inégalités se résume ainsi. On se ramène à $\alpha > \beta = 1$. Le cas $p = \infty$ est immédiat en utilisant des considérations élémentaires sur les tentes. Les mêmes considérations entraînent les inégalités pour les atomes et donc pour $0 < p \leqslant 1$. Enfin, on interpole les résultats déjà obtenus avec l'identité $\|g\|_{T^{2,2}_\alpha} = \alpha^{n/2}\|g\|_{T^{2,2}_1}$. Les exemples montrant l'optimalité sont des indicatrices supportées dans une tente et nulles près du bord. La preuve complète se trouve plus loin.

On peut citer plusieurs corollaires. Le premier donne une comparaison précise avec la norme $\|Vg\|_p$ de la fonctionnelle de Littlewood–Paley verticale classique $Vg(x) = (\int_0^\infty |g(t,x)|^2 \frac{dt}{t})^{\frac{1}{2}}$. On obtient $\|Vg\|_p \leqslant C\|g\|_{T^{p,2}_1}$ si $p \leqslant 2$, et $\|Vg\|_p \geqslant C\|g\|_{T^{p,2}_1}$ si $p \geqslant 2$. Le résultat est connu par d'autres méthodes (voir [9,1]). Il se retrouve en observant que pour $\alpha \to 0$, $\alpha^{-n/2}A^{(\alpha)}g(x)$ converge vers $Vg(x)$ pour des fonctions $g \in C^\infty$ à support compact et en utilisant notre résultat. Noter que les inégalités inverses à celles ci-dessus sont toutes fausses si $p \neq 2$ (voir [1]).

Un deuxième corollaire est l'inégalité

$$\left\| \left(\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{|x-y|+t} \right)^{n\lambda} \left| g(t,y) \right|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \right)^{\frac{1}{2}} \right\|_p \leqslant C(n,p,\lambda) \left\| A^{(1)} g \right\|_p$$

pour $0 < p < \infty$ et $\lambda > \max(2/p, 1)$ et toute fonction g. En appliquant cette inégalité à $g(t,y) = t\nabla u(t,y)$, u étant l'extension harmonique d'une distribution appropriée u_0, cela donne une démonstration simplifiée de l'inégalité de Stein (Theorem 2, Chap. IV, [9]) portant sur la grande fonction carrée une fois connue celle-sur la fonction d'aire de Lusin.

1. Introduction and main results

Let $B(x,t)$ denote an open ball centered at $x \in \mathbb{R}^n$ with radius $t > 0$. Define for a locally square integrable function $g(t,y)$, $(t,y) \in \mathbb{R}^{n+1}_+$, for $\alpha > 0$ and $x \in \mathbb{R}^n$,

$$A^{(\alpha)} g(x) := \left(\int_{\mathbb{R}^{n+1}_+} \frac{1_{B(x,\alpha t)}(y)}{t^n} \left| g(t,y) \right|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t} \right)^{\frac{1}{2}}, \tag{1}$$

and for $0 < p < \infty$, say that $g \in T^{p,2}_\alpha$ if

$$\|g\|_{T^{p,2}_\alpha} := \left\| A^{(\alpha)} g \right\|_p < \infty.$$

This space was introduced in [4]. As sets the spaces for a given p are the same and the norms (or quasi-norms if $p < 1$) are equivalent: whenever $\alpha, \beta > 0$, one has

$$\left\| A^{(\alpha)} g \right\|_p \sim \left\| A^{(\beta)} g \right\|_p. \tag{2}$$

For $p = \infty$, the limiting space is defined with a Carleson measure condition. Let $T_\alpha B$ be the tent with aperture α above the open ball $B = B(x,r)$, *i.e.*, the set of (t,y) such that $0 < t < r/\alpha$ and $y \in B(x, r-\alpha t)$. We define $\|g\|_{T^{\infty,2}_\alpha}$ as the infimum of $C \geqslant 0$ such that for all ball B,

$$\int_{T_\alpha B} \left| g(t,y) \right|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t} \leqslant C^2 \frac{|B|}{\alpha^n}. \tag{3}$$

Again, the spaces $T^{\infty,2}_\alpha$ are the same, the norms are equivalent and the isometry property holds.

To explain the choice of the normalization in (3), we remark that for $p = \infty$ included, $g(t,y) \mapsto h(t,y) := \alpha^{n/2} g(t/\alpha, y)$ is an isometry between $T^{p,2}_1$ and $T^{p,2}_\alpha$ equipped with their respective (quasi-)norms.

It is shown in [4] that the spaces $T^{p,2}_1$, $0 < p \leqslant \infty$, interpolate by the complex method and the real method. The same results hold for $T^{p,2}_\alpha$ for fixed α, with constants (*i.e.*, the constants in the equivalence of norms between $T^{p,2}_\alpha$ and the interpolated space to which it is equal) independent of α by using the isometry property.

Motivated by an intensive usage of tent spaces in the development of new Hardy spaces associated to operators with Gaffney–Davies estimates first made in [2,7], and also by the study of maximal regularity on tent spaces towards applications for parabolic PDE's ([3], and some more work in progress), it became interesting to know the sharp dependence of the bounds in (2) with respect to α, β. The L^2 bound is immediate by Fubini's theorem: $\|g\|_{T^{2,2}_\alpha} = (\alpha/\beta)^{n/2}\|g\|_{T^{2,2}_\beta}$. For $p \neq 2$, the argument in [4], originating from [6], does not give optimal dependence. The inequality

$$\|g\|_{T^{p,2}_\alpha} \leqslant C(1+\log\alpha)\alpha^{n/\tau}\|g\|_{T^{p,2}_1},$$

for $1 < p < \infty$ and $\alpha \geqslant 1$, where $\tau = \min(p,2)$ and C depends only on n and p, is proved in [8] as a special case of a Banach space valued result, and, moreover, the polynomial growth $\alpha^{n/\tau}$ is shown to be optimal for such an inequality to hold. The restriction $p > 1$ occurs in this argument because the UMD property is required and a maximal inequality is used. Note that even the L^2 bounds is not immediate in a Banach (non-Hilbert) space valued context. In discussion with T. Hytönen, we convinced ourselves that the logarithmic factor is not produced by this argument in the scalar case when $p \geqslant 2$. Still, this argument is quite involved and elimination of the logarithm in the $p < 2$ situation was unclear.

Here we give the sharp lower and upper bounds for (2) in the scalar case by a very simple argument. Define $\underline{h}(p,\alpha) = \min(\alpha^{-n/2}, \alpha^{-n/p})$, $\bar{h}(p,\alpha) = \max(\alpha^{-n/2}, \alpha^{-n/p})$. Note that $\underline{h}(p,\alpha) = \alpha^{-n/p}$ if $(\alpha-1)(p-2) \geqslant 0$ and $\underline{h}(p,\alpha) = \alpha^{-n/2}$ if $(\alpha-1)(p-2) \leqslant 0$, and inversely for $\bar{h}(p,\alpha)$.

Theorem 1.1. *Let $0 < p \leqslant \infty$ and $\alpha, \beta > 0$. There exist constants $C, C' > 0$ depending on n, p only, such that for any locally square integrable function g,*

$$C\underline{h}(p,\alpha/\beta)\|g\|_{T^{p,2}_\alpha} \leqslant \|g\|_{T^{p,2}_\beta} \leqslant C'\bar{h}(p,\alpha/\beta)\|g\|_{T^{p,2}_\alpha}.$$

Moreover, the dependence in α/β is best possible in the sense that this growth is attained.

In particular, for $\alpha > 1$, $\tau = \min(2,p)$ and $\sigma = \max(2,p)$, one has

$$\|g\|_{T^{p,2}_\alpha} \leqslant C\alpha^{n/\tau}\|g\|_{T^{p,2}_1}, \tag{4}$$

$$\|g\|_{T^{p,2}_1} \leqslant C\alpha^{-n/\sigma}\|g\|_{T^{p,2}_\alpha}. \tag{5}$$

The second one improves the obvious bound $\|g\|_{T^{p,2}_1} \leqslant \|g\|_{T^{p,2}_\alpha}$. By symmetry using the relation $\underline{h}(p,\alpha)^{-1} = \bar{h}(p,\alpha^{-1})$ and scale invariance, all cases reduce to (4) and (5) with $\alpha > 1$.

Corollary 1.2. *Let $0 < p < \infty$. There is a constant $0 < C < \infty$ depending on n, p only such that for any locally square integrable g, if $Vg(x) = (\int_0^\infty |g(t,x)|^2 \frac{dt}{t})^{\frac{1}{2}}$,*

$$\|Vg\|_p \leqslant C\|g\|_{T^{p,2}_1}, \quad \text{if } p \leqslant 2, \qquad \|Vg\|_p \geqslant C\|g\|_{T^{p,2}_1}, \quad \text{if } p \geqslant 2.$$

The corollary was proved by a different method in [1] when $p < 2$ and the $p > 2$ case dates back to [9]. The opposite inequalities are false if $p \neq 2$. Starting from Theorem 1.1, the proof is a mere application of Lebesgue differentiation theorem when $\alpha \to 0$ for $\alpha^{-n/2}A^{(\alpha)}g(x)$ converges to $Vg(x)$ assuming g smooth with compact support. This assumption is easily removed.

Corollary 1.3. *If $0 < p < \infty$ and $\lambda > \max(2/p, 1)$, for any locally square integrable g,*

$$\left\| \left(\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{|x-y|+t} \right)^{n\lambda} |g(t,y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}} \right\|_p \leqslant C(n,p,\lambda)\left\|A^{(1)}g\right\|_p.$$

The left-hand side equals the grand square function of Stein when $g(t,y) = t\nabla u(t,y)$, u being the harmonic extension of a suitable distribution u_0 on $\mathbb{R}^n$. Hence, for all $0 < p < \infty$ and $\lambda > \max(2/p, 1)$, it is dominated in L^p by $\|A^{(1)}g\|_p$ which is the L^p norm of the area functional of Lusin defined from u_0. However, it is known from Stein–Weiss' theory that $A^{(1)}g \in L^p(\mathbb{R}^n)$ if and only if $\frac{n-1}{n} < p < \infty$ and u_0 belongs to the Hardy space $H^p(\mathbb{R}^n)$ (see [9]). This gives a simple proof of Theorem 2, Chap. IV in [9]. The lower exponent $\frac{n-1}{n}$ is only due to the choice of the extension. Using an extension by convolution with $t^{-n}\varphi(x/t)$ with $\varphi \in C_0^\infty(\mathbb{R}^n)$ having all vanishing moments but the one of order 0, $\frac{n-1}{n}$ becomes 0 by the results in [6]. At $\lambda = 2/p$ and $p < 2$, a weak type inequality is plausible, the Lorentz norm $L^{p,\infty}$ replacing the Lebesgue norm L^p in the left-hand side. It would give a simple proof of the weak type (p,p) result of Fefferman [5] for Stein's grand square function. We leave this open.

The proof of the corollary is easy by splitting the upper half space according to $|x-y|/t$ compared to powers 2^k, $k \geqslant 0$, and one obtains

$$\left(\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{|x-y|+t} \right)^{n\lambda} |g(t,y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}} \leqslant C(n,\lambda)\sum_{k\geqslant 0} 2^{-kn\lambda/2}A^{(2^{k+1})}g(x).$$

It remains to use $\|A^{(\alpha)}g\|_p \leqslant C\alpha^{n/\tau}\|A^{(1)}g\|_p$ for $\alpha = 2^{k+1}$ in appropriate arguments for $p \geqslant 1$ or $p \leqslant 1$.

The proof of Theorem 1.1 is an easy matter using in part atomic theory for tent spaces, again proved in [4]. Recall that for $0 < p \leqslant 1$, the tent space $T_1^{p,2}$ has an atomic decomposition: A $T_1^{p,2}$ atom is a function $a(t,x)$ supported in a tent T_1B with the estimate

$$\int\limits_{T_1B} \left|a(t,x)\right|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \leqslant |B|^{-(\frac{2}{p}-1)}.$$

There is a constant $C = C(n,p) > 0$ such that $\|a\|_{T_1^{p,2}} \leqslant C$. Any $T_1^{p,2}$ function g can be represented as a series $g = \sum \lambda_j a_j$ where a_j is a $T_1^{p,2}$ atom and $\sum |\lambda_j|^p \sim \|g\|^p_{T_1^{p,2}}$. By the isometry property, a $T_\alpha^{p,2}$ atom is a function $A(u,x)$ supported a tent $T_\alpha B$ with the estimate

$$\int\limits_{T_\alpha B} \left|A(u,x)\right|^2 \frac{\mathrm{d}x\,\mathrm{d}u}{u} \leqslant \alpha^{-n}|B|^{-(\frac{2}{p}-1)}$$

and the decomposition theorem holds in $T_\alpha^{p,2}$.

2. Proof of Theorem 1.1

Recall that we may assume $\alpha > \beta = 1$ and it is enough to prove (4) and (5). Fix $p = \infty$ first. Let $\|g\|_{T_1^{\infty,2}} = 1$ and B be a ball. As $T_\alpha B \subset T_1 B$,

$$\int\limits_{T_\alpha B} \left|g(t,x)\right|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \leqslant \int\limits_{T_1 B} \left|g(t,x)\right|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \leqslant |B| = \alpha^n \frac{|B|}{\alpha^n}.$$

Hence, $\|g\|_{T_\alpha^{\infty,2}} \leqslant \alpha^{n/2}$. This shows $\|g\|_{T_\alpha^{\infty,2}} \leqslant \alpha^{n/2}\|g\|_{T_1^{\infty,2}}$ for all g.

Let $\|g\|_{T_\alpha^{\infty,2}} = 1$ and B be a ball. As $T_1 B \subset T_\alpha(\alpha B)$ where αB is the ball concentric with B dilated by α,

$$\int\limits_{T_1 B} \left|g(t,x)\right|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \leqslant \int\limits_{T_\alpha(\alpha B)} \left|g(t,x)\right|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \leqslant \frac{|\alpha B|}{\alpha^n} = |B|.$$

Hence, $\|g\|_{T_1^{\infty,2}} \leqslant 1$ and $\|g\|_{T_1^{\infty,2}} \leqslant \|g\|_{T_\alpha^{\infty,2}}$ for all g.

Fix now $p \leqslant 1$. Let B be a ball and a be a $T_1^{p,2}$ atom supported in a tent $T_1 B$. As $T_1 B \subset T_\alpha(\alpha B)$, we have a is supported in $T_\alpha(\alpha B)$ and

$$\int\limits_{T_\alpha(\alpha B)} \left|a(t,x)\right|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} = \int\limits_{T_1 B} \left|a(t,x)\right|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \leqslant |B|^{-(\frac{2}{p}-1)} = \alpha^{2n/p}\big(\alpha^{-n}|\alpha B|^{-(\frac{2}{p}-1)}\big).$$

Thus $\alpha^{-n/p}a$ is a $T_\alpha^{p,2}$ atom. An atomic decomposition of any element of $T_1^{p,2}$ is up to multiplication by $\alpha^{-n/p}$ an atomic decomposition in $T_\alpha^{p,2}$, proving $\|g\|_{T_\alpha^{p,2}} \leqslant C(n,p)\alpha^{n/p}\|g\|_{T_1^{p,2}}$.

Next, let B be a ball and a be a $T_\alpha^{p,2}$ atom supported in a tent $T_\alpha B$. As $T_\alpha B \subset T_1 B$, we have a is supported in $T_1 B$ and

$$\int\limits_{T_1 B} \left|a(t,x)\right|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} = \int\limits_{T_\alpha B} \left|a(t,x)\right|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \leqslant \alpha^{-n}|B|^{-(\frac{2}{p}-1)}.$$

Thus $\alpha^{n/2}a$ is a $T_1^{p,2}$ atom. As above, we conclude that $\|g\|_{T_1^{p,2}} \leqslant C(n,p)\alpha^{-n/2}\|g\|_{T_\alpha^{p,2}}$.

For $1 < p < 2$ and $2 < p < \infty$, we conclude by interpolation with the $p = 2$ equality $\|g\|_{T_\alpha^{2,2}} = \alpha^{n/2}\|g\|_{T_1^{2,2}}$. We have shown (4) and (5).

The sharpness of the bounds is seen by saturating these inequalities. Fix $\alpha > 1$ large. Let B be the unit ball. Set $a_1(t,y) = 1_{T_1 B}(t,y)1_{[1/2,1]}(t)$. It is easy to see that $\|a_1\|_{T_1^{p,2}} \sim 1$. Now, we have that $A^{(\alpha)}a_1$ has support equal to $\overline{B}(0,\alpha)$, is bounded by a constant $c(n) > 0$ and equal to that constant on the ball $B(0, \frac{\alpha+1}{2})$. Thus $\|a_1\|_{T_\alpha^{p,2}} \sim \alpha^{n/p}$. This proves that (4) is optimal when $p \leqslant 2$ and (5) is optimal when $p \geqslant 2$. Next, let $a_2(t,y) = a_1(\alpha t, y)$. By scaling $\|a_2\|_{T_\alpha^{p,2}} = \alpha^{n/2}\|a_1\|_{T_1^{p,2}} \sim \alpha^{n/2}$. Now, $A^{(1)}a_2$ is supported in $\overline{B}$, bounded by a constant $c(n) > 0$ and equal to that constant on $B(0, \frac{\alpha-1}{2\alpha})$. Thus $\|a_2\|_{T_1^{p,2}} \sim 1$. This proves that (4) is optimal when $p \geqslant 2$ and (5) is optimal when $p \leqslant 2$. $\square$

References

[1] P. Auscher, S. Hofmann, J.-M. Martell, Vertical versus conical square functions, preprint, arXiv:1012.4184.
[2] P. Auscher, A. McIntosh, E. Russ, Hardy spaces of differential forms and Riesz transforms on Riemannian manifolds, J. Geom. Anal. 18 (2008) 192–248.
[3] P. Auscher, S. Monniaux, P. Portal, The maximal regularity operator on tent spaces, arXiv:1011.1748. Comm. Pure Appl. Anal., in press.
[4] R. Coifman, Y. Meyer, E.M. Stein, Some new function spaces and their applications to harmonic analysis, J. Funct. Anal. 62 (1985) 304–335.
[5] C. Fefferman, Inequalities for strongly singular convolution operators, Acta Math. 124 (1970) 9–36.
[6] C. Fefferman, E.M. Stein, H^p spaces of several variables, Acta Math. 129 (1972) 137–193.
[7] S. Hofmann, S. Mayboroda, Hardy and BMO spaces associated to divergence form elliptic operators, Math. Ann. 344 (1) (2009) 37–116.
[8] T. Hytönen, J. van Neerven, P. Portal, Conical square function estimates in UMD Banach spaces and applications to H^∞-functional calculi, J. Analyse Math. 106 (2008) 317–351.
[9] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Mathematical Series, vol. 30, Princeton University Press, Princeton, N.J., 1970.