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On wave propagation in the Anti-de Sitter cosmology

Sur la propagation des ondes dans la cosmologie d'Anti-de Sitter

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ABSTRACT

We investigate the Klein–Gordon equation on the Poincaré patch of the 5-dimensional Anti-de Sitter universe. Despite the loss of the global hyperbolicity, the Cauchy problem is well-posed in the space of the finite energy data. We establish several results of asymptotic behaviours. We consider also the cosmological model of the Minkowski brane with a negative tension, for which we solve the mixed problem.

RÉSUMÉ

Nous étudions l'équation de Klein–Gordon sur la variété de Poincaré Anti-de Sitter de dimension 5. En dépit de la perte d'hyperbolicité globale, le problème de Cauchy est bien posé dans un cadre d'énergie finie. On établit plusieurs résultats de comportements asymptotiques. Nous considérons aussi le modèle cosmologique de la membrane de Minkowski de tension négative, pour laquelle nous résolvons le problème mixte.

Version française abrégée

L'équation de Klein–Gordon dans la carte de Poincaré de l'espace-temps Anti-de Sitter $\mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times]0,\infty[_z$, $\mathrm{d}s^2 = z^{-2}(\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2 - \mathrm{d}z^2)$, a la forme

$$\left(\partial_t^2 - \Delta_{\mathbf{x}} - \partial_z^2 + \frac{\lambda^2 - \frac{1}{4}}{z^2}\right)\Phi = 0. \tag{1}$$

$\lambda = 2$ pour les fluctuations du tenseur métrique et $\lambda = 1$ pour les champs vectoriels électromagnétiques. Pour $\lambda > 0$, le problème de Cauchy est bien posé sous la contrainte d'énergie finie

$$\int_{\mathbb{R}^3}\int_0^\infty \left|\nabla_{t,\mathbf{x},z}\Phi(t,\mathbf{x},z)\right|^2 + \frac{\lambda^2 - \frac{1}{4}}{z^2}\left|\Phi(t,\mathbf{x},z)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}z < \infty$$

et la solution est une superposition continue de champs de Klein–Gordon ϕ_m de masse m sur l'espace de Minkowski $\mathbb{R}^{1+3}$:

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$$\Phi(t,\mathbf{x},z)=\lim_{M\to\infty}\int_0^M \phi_m(t,\mathbf{x})\sqrt{mz}\,J_\lambda(mz)\,\mathrm{d}m.$$

On établit une estimation L^1-L^∞ de décroissance en temps pour les données régulières :

$$\left\|z^{-\frac{1}{2}-\lambda}\Phi(t,.)\right\|_{L^\infty(\mathbb{R}^3\times]0,\infty[)}\lesssim |t|^{-\frac{3}{2}}.$$

Quand $\lambda=\nu/2$, $\nu\in\mathbb{N}^*$, Φ s'exprime en fonction d'une solution de l'équation des ondes dans un espace de Minkowski de haute dimension. On en déduit des inégalités globales de Strichartz, une décroissance uniforme plus rapide en $|t|^{-\lambda-2}$, l'existence d'une lacune et de l'équipartition de l'énergie à temps fini pour les données à support compact, et un résultat de réflection des singularités à l'horizon $z=0$.

On considère aussi le modèle de cosmologie branaire de la membrane de Minkowski de tension négative, qui est la frontière $z=1$ du sous-domaine $\mathcal{B}=\mathbb{R}_t\times\mathbb{R}^3_{\mathrm{x}}\times]0,1[_z$. On résoud le problème mixte pour l'équation (1) dans $\mathcal{B}$ avec la condition de Robin sur la membrane

$$\partial_z\Phi(t,\mathbf{x},1)+\frac{3}{2}\Phi(t,\mathbf{x},1)=0,\quad t\in\mathbb{R},\ \mathbf{x}\in\mathbb{R}^3.$$

Les solutions faibles sont des superpositions discrètes de champs de Klein–Gordon ϕ_n de masse $\lambda_n\to\infty$ sur l'espace de Minkowski $\mathbb{R}^{1+3}$:

$$\Phi(t,\mathbf{x},z)=\sum_{n=0}^{\infty}\phi_n(t,\mathbf{x})u_n(z).$$

Pour un ensemble dense de données initiales, la solution satisfait l'estimation :

$$\left|\Phi(t,\mathrm{x},z)\right|\lesssim\left(|t|+|\mathrm{x}|\right)^{-\frac{3}{2}}z^{\lambda+\frac{1}{2}}.$$

1. The Klein–Gordon equation on the Poincaré–Anti-de Sitter space–time

The Poincaré patch of the Anti-de Sitter universe AdS^5 is the Lorentzian manifold

$$\mathcal{P}:=\mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z,\ g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu=z^{-2}\left(\mathrm{d}t^2-\mathrm{d}\mathbf{x}^2-\mathrm{d}z^2\right).\tag{2}$$

The boundary of this universe, that is located at $z=0$, is time-like and many null geodesics hit this horizon, so $\mathcal{P}$ is not globally hyperbolic and the well-posedness of the Cauchy problem for the hyperbolic equations of the fields theory is doubtful. This question has been investigated in [1] and [6] for the analogous case of the whole universal covering of the Anti-de Sitter space–time. The main aim of this Note consists in understanding the role of this horizon for the propagation of the scalar waves in $\mathcal{P}$ and their asymptotic behaviour at the infinity. The detailed proofs can be found in [3]. Given $\alpha\in\mathbb{R}$, we consider the Klein–Gordon equation

$$|g|^{-\frac{1}{2}}\partial_\mu\left(|g|^{\frac{1}{2}}g^{\mu\nu}\partial_\nu\right)u+\alpha u=0.$$

When we put $\Phi=:z^{-\frac{3}{2}}u$ and $\lambda^2=\alpha+4$, this equation takes a very simple form where the loss of the global hyperbolicity is expressed by the singularity of a Cartesian potential at the horizon which acts as a repulsive boundary:

$$\left(\partial_t^2-\Delta_{\mathbf{x}}-\partial_z^2+\frac{\lambda^2-\frac{1}{4}}{z^2}\right)\Phi=0.\tag{3}$$

$\lambda=2$ for the tensor type gravitational fluctuations and $\lambda=1$ for the vector type electromagnetic waves of the vector type gravitational waves. To solve the Cauchy problem for the finite energy waves, we put:

$$\|\Phi\|^2_{BL^1_0}:=\int_{\mathbb{R}^3}\int_0^\infty\left|\nabla_{\mathbf{x},z}\Phi(\mathbf{x},z)\right|^2+\frac{\lambda^2-\frac{1}{4}}{z^2}\left|\Phi(\mathbf{x},z)\right|^2\mathrm{d}\mathbf{x}\,\mathrm{d}z.$$

The Hardy inequality assures that if $\lambda>0$ this integral defines a norm on $C_0^\infty(\mathbb{R}^3\times]0,\infty[)$ of which we denote $BL^1_0(\mathbb{R}^3\times]0,\infty[)$ the closure. The requirement of the finiteness of the energy, imposes implicitly the Dirichlet boundary condition on the horizon.

(1) The global Cauchy problem is well-posed: we prove by a classic spectral method that for any $\lambda>0$, given $\Phi_0\in BL^1_0(\mathbb{R}^3\times]0,\infty[)$ and $\Phi_1\in L^2(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)$, there exists a unique solution $\Phi\in C^0(\mathbb{R}_t;BL^1_0)$ of (3) satisfying $\partial_t\Phi\in C^0(\mathbb{R}_t;L^2)$ and the Cauchy condition $\Phi(0,.)=\Phi_0(.)$, $\partial_t\Phi(0,.)=\Phi_1(.)$. Moreover the energy $\|\Phi(t,.)\|^2_{BL^1_0}+\|\partial_t\Phi(t,.)\|^2_{L^2}$ is conserved, and if $\Phi_0(\mathbf{x},z)=\Phi_1(\mathbf{x},z)=0$ when $|\mathbf{x}|\geqslant R$ or $|z|\geqslant R$, then $\Phi(t,\mathbf{x},z)=0$ when $|\mathbf{x}|\geqslant R+|t|$ or $|z|\geqslant R+|t|$.

(2) In brane cosmology it is important to express the fields propagating in the Anti-de Sitter universe, as a Kaluza–Klein tower, by decoupling the familiar variables (t, x) from the space variable of depth z. We show that

$$\Phi(t,\mathbf{x},z)=\lim_{M\to\infty}\int_0^M \phi_m(t,\mathbf{x})\sqrt{mz}\,J_\lambda(mz)\,\mathrm{d}m \quad \text{in } C^0\big(\mathbb{R}_t;BL_0^1\big),$$
$$\partial_t\Phi(t,\mathbf{x},z)=\lim_{M\to\infty}\int_0^M \partial_t\phi_m(t,\mathbf{x})\sqrt{mz}\,J_\lambda(mz)\,\mathrm{d}m \quad \text{in } C^0\big(\mathbb{R}_t;L^2\big), \tag{4}$$

and ϕ_m is solution for almost all $m>0$, of $\partial_t^2\phi_m-\Delta_{\mathbf{x}}\phi_m+m^2\phi_m=0$ in $\mathbb{R}^{1+3}$, satisfying for any $T>0$:

$$\phi_m\in L^2\big(]0,\infty[_m;C^0\big([-T,T]_t;\dot H^1\big(\mathbb{R}^3_{\mathbf{x}}\big)\big)\big)\cap L^2_{loc}\big(]0,\infty[_m;C^0\big([-T,T]_t;H^1\big(\mathbb{R}^3_{\mathbf{x}}\big)\big)\big),$$
$$\partial_t\phi_m\in L^2\big(]0,\infty[_m;C^0\big([-T,T]_t;L^2\big(\mathbb{R}^3_{\mathbf{x}}\big)\big)\big).$$

Moreover, $\|\Phi_0\|^2_{BL_0^1}+\|\Phi_1\|^2_{L^2}=\int_0^\infty\|\nabla_{t,\mathbf{x}}\phi_m(t)\|^2_{L^2(\mathbb{R}^3)}+m^2\|\phi_m(t)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}m$.

(3) We can deduce from (4) some L^1-L^∞ estimate with a control near the horizon, depending on the mass:

$$\big\|z^{-\lambda-\frac12}\Phi(t,.)\big\|_{L^\infty(\mathbb{R}^3\times]0,\infty[)}\lesssim|t|^{-\frac32}\sum_{j=0,1}\sum_{|\alpha|+j\leqslant3}\big\|\partial_{\mathrm{x}}^\alpha\Phi_j\big\|_{L^1}+\left\|\partial_{\mathrm{x}}^\alpha\left(-\partial_z^2+\frac{\lambda^2-\frac14}{z^2}\right)^{[\frac{\lambda+3-|\alpha|-j}{2}]+1}\Phi_j\right\|_{L^1}.$$

The rate of time decay equals to that of the massive waves in the Minkowski space–time $\mathbb{R}^{1+3}$. It will be significantly improved in the next part for some values of the mass.

2. The case of the mass $\lambda=\nu/2$, $\nu\in\mathbb{N}^*$

The equations for the gravitational fluctuations or the electromagnetic perturbations belong to a large class for which $\lambda=\nu/2$, $\nu\in\mathbb{N}^*$. For these values of λ, the finite energy solutions are closely linked to the finite energy solutions of the free wave equation on the Minkowski space–time with a higher dimension. The method rests on a very simple observation: we can consider the fifth space-like dimension $z>0$, as the radial coordinate of some Euclidean high-dimensional space $\mathbb{R}^N_{\mathbf{z}}$, $N\geqslant2$, i.e. $z=|\mathbf{z}|$. We denote $Y_{l,m}$ the generalized spherical harmonics that form a orthonormal basis of $L^2(S^{N-1})$ of eigenfunctions of the Laplace–Beltrami operator satisfying $\Delta_{S^{N-1}}Y_{l,m}=-l(l+N-2)Y_{l,m}$. Here $l,m\in\mathbb{N}$ and m is bounded by the dimension of the space of harmonic homogeneous polynomials of degree l in N variables. As a consequence of (4) and $-\Delta_{\mathrm{z}}[|\mathrm{z}|^{-\frac N2+1}J_\lambda(m|\mathrm{z}|)]=m^2|\mathrm{z}|^{-\frac N2+1}J_\lambda(m|\mathrm{z}|)$ in $\mathbb{R}^N_z$ with $\lambda=\frac N2+l-1$, we show that if Φ is a finite energy solution of (3), then $\Psi(t,\mathbf{x},\mathbf{z})=|\mathbf{z}|^{-\frac{N-1}{2}}\Phi(t,\mathbf{x},|\mathbf{z}|)Y_{l,m}(\frac{\mathbf{z}}{|\mathbf{z}|})$ is a finite energy solution of $\partial_t^2\Psi-\Delta_{\mathbf{x},\mathbf{z}}\Psi=0$ in the whole Minkowski space $\mathbb{R}\times\mathbb{R}^{3+N}$. In this case, we can easily deduce the properties of the finite energy solutions from the properties of the free waves in the Minkowski space–time.

(1) As regards the propagation of the singularities, the time-like horizon acts like a perfectly reflecting mirror: if $(t,\mathbf{x},z;\tau,\boldsymbol{\xi},\zeta)\in WF(\Phi)$ then for all $\sigma\in\mathbb{R}$, $\sigma\zeta\neq z$ we have

$$\big(t+\sigma\tau,\mathbf{x}-\sigma\boldsymbol{\xi},|z-\sigma\zeta|;\tau,\boldsymbol{\xi},(z-\sigma\zeta)|z-\sigma\zeta|^{-1}\zeta\big)\in WF(\Phi).$$

A. Vasy has proved this result for the D'Alembertian [8].

(2) We obtain some L^2-L^∞ estimates from the sharp results of [5]. We introduce suitable weighted Sobolev spaces in the spirit of Y. Choquet-Bruhat and D. Christodoulou [4], to impose some constraints at the space infinity and at the horizon $z=0$. Given an integer $s\geqslant0$ and a real δ, we define the space $H_0^{s,\delta}(\mathbb{R}^3_{\mathrm{x}}\times]0,\infty[_z)$ as the completion of $C_0^\infty(\mathbb{R}^3_{\mathrm{x}}\times]0,\infty[_z)$ for the norm:

$$\|\Phi\|^2_{H_0^{s,\delta}}:=\sum_{|\alpha|\leqslant s}\sum_{k=0}^{s-|\alpha|}\sum_{b=0}^{k}\big\|\big(z^k+z^{b-k}\big)\big(1+|\mathrm{x}|^2+z^2\big)^{\frac{\delta+|\alpha|+k}{2}}\partial_{\mathrm{x}}^\alpha\partial_z^b\Phi\big\|^2_{L^2(\mathbb{R}^3_{\mathrm{x}}\times]0,\infty[_z)}.$$

Then for all $\epsilon>0$ there exists $C(\epsilon)>0$ such that the finite solutions of (3) satisfy:

$$\big(1+|t|+|\mathrm{x}|+z+\big|t^2-|\mathrm{x}|^2-z^2\big|\big)^{\frac\nu2+2}z^{-\frac{\nu+1}{2}}\big|\Phi(t,\mathrm{x},z)\big|\leqslant C(\epsilon)\Big(\|\Phi_0\|_{H_0^{[\frac{\nu+7}{2}],\frac{\nu+3}{2}+\epsilon}}+\|\Phi_1\|_{H_0^{[\frac{\nu+5}{2}],\frac{\nu+5}{2}+\epsilon}}\Big).$$

We remark that the rate of the time decay in $\mathcal{P}$ increases with the mass of the field, unlike the case of the Minkowski space.

(3) We prove also Strichartz type estimates in which the energy allows to control some global L^p norms with a strong constraint at the horizon:

$$\left\| z^{(\nu+1)(\frac{1}{r}-\frac{1}{2})}\Phi \right\|_{L^q(\mathbb{R}_t;L^r(\mathbb{R}^3_{\mathrm{x}}\times]0,\infty[_z))} \lesssim \left(\|\Phi_0\|_{BL^1_0}+\|\Phi_1\|_{L^2}\right)$$

when $2 \leqslant q$, $1/q+(\nu+5)/r=(\nu+3)/2$, $1/q+(\nu+4)/2r \leqslant (\nu+4)/4$. Moreover, if $\nu=2k$, $k\in\mathbb{N}^*$ and $2\leqslant q$, $1/q+5/r=3/2$, $1/q+2/r\leqslant 1$, then

$$\left\| z^{(\frac{1}{r}-\frac{1}{2})}\Phi \right\|_{L^q(\mathbb{R}_t;L^r(\mathbb{R}^3_{\mathrm{x}}\times]0,\infty[_z))} \lesssim \left(\|\Phi_0\|_{BL^1_0}+\|\Phi_1\|_{L^2}\right),$$

and if $\nu=2k+1$, $k\in\mathbb{N}$, and $2\leqslant q$, $1/q+6/r=2$, $1/q+5/2r\leqslant 5/4$ then

$$\left\| z^{(\frac{2}{r}-1)}\Phi \right\|_{L^q(\mathbb{R}_t;L^r(\mathbb{R}^3_{\mathrm{x}}\times]0,\infty[_z))} \lesssim \left(\|\Phi_0\|_{BL^1_0}+\|\Phi_1\|_{L^2}\right).$$

(4) The Huygens Principle fails for Eq. (3) since the number of the space dimensions is even and the Hadamard's criterion is not satisfied. Nevertheless we prove some unexpected results: the existence of a lacuna, and also the equipartition of the energy at finite time when the initial data are compactly supported and ν is even. If $\Phi_0=\Phi_1=0$ when $|\mathbf{x}|^2+z^2\geqslant R^2$ for some $R>0$, then $\Phi(t,\mathbf{x},z)=0$ when $|\mathbf{x}|^2+z^2\leqslant(|t|-R)^2$ and $|t|\geqslant R$, and the potential and kinetic energies are equal:

$$\int_{\mathbb{R}^3}\int_0^\infty \left|\nabla_{\mathbf{x},z}\Phi(t,\mathbf{x},z)\right|^2+\frac{\lambda^2-\frac{1}{4}}{z^2}\left|\Phi(t,\mathbf{x},z)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}z=\int_{\mathbb{R}^3}\int_0^\infty \left|\partial_t\Phi(t,\mathbf{x},z)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}z.$$

3. L^2 solutions in brane cosmology

In brane cosmology, the Minkowski space–time $\mathbb{R}_t\times\mathbb{R}^3_{\mathrm{x}}$ is considered as a brane that is the boundary $\mathbb{R}_t\times\mathbb{R}^3_{\mathrm{x}}\times\{z=1\}$ of a part $\mathcal{B}\subset\mathcal{P}$, of which the choice depends on the tension of this brane (see [7]). The RS2 Randall–Sundrum model investigated in [2], deals with the Minkowski brane with a positive tension that is the boundary of $\mathbb{R}_t\times\mathbb{R}^3_{\mathrm{x}}\times]1,\infty[_z$. In this part we consider the case of the Minkowski brane with a negative tension. In this case $\mathcal{B}=\mathbb{R}_t\times\mathbb{R}^3_{\mathrm{x}}\times]0,1[_z$ and we have to study the Klein–Gordon equation (3) in $\mathcal{B}$. The boundary condition on the brane is associated to the Z_2 symmetry (see [7]) that yields to the Robin condition:

$$\partial_z\Phi(t,\mathbf{x},1)+\frac{3}{2}\Phi(t,\mathbf{x},1)=0,\quad t\in\mathbb{R},\ \mathbf{x}\in\mathbb{R}^3. \tag{5}$$

To solve the mixed problem when $\lambda>0$, we introduce the Hilbert space

$$W^1=\left\{\phi\in H^1\left(\mathbb{R}^3_{\mathbf{x}}\times]0,1[_z\right);\ \phi(\mathbf{x},0)=0\right\},\qquad \|\phi\|^2_{W^1}=\int_{\mathbb{R}^3}\int_0^1 |\nabla_{\mathbf{x},z}\phi|^2+\frac{\lambda^2-\frac{1}{4}}{z^2}|\phi|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}z+\frac{3}{2}\int_{\mathbb{R}^3}\phi(\mathbf{x},1)\,\mathrm{d}\mathbf{x}.$$

Thanks to the Hardy inequality, $\|.\|_{W^1}$ is a norm on W^1 when $\lambda>0$, that is equivalent to the usual H^1-norm. To take account of the boundary condition, we introduce also the space

$$W^2=\left\{\phi\in W^1;\ \Delta_{\mathbf{x},z}\phi-\frac{\lambda^2-\frac{1}{4}}{z^2}\phi\in L^2,\ \partial_z\phi(\mathbf{x},1)+\frac{3}{2}\phi(\mathbf{x},1)=0\right\},$$
$$\|\phi\|^2_{W^2}=\|\phi\|^2_{W^1}+\left\|\Delta_{\mathbf{x},z}\phi-\frac{\lambda^2-\frac{1}{4}}{z^2}\phi\right\|^2_{L^2}.$$

The boundary condition on the brane makes sense since $W^2\subset C^1(]0,1]_z;H^{-\frac{1}{2}}(\mathbb{R}^3_{\mathbf{x}}))$. To construct the weak solutions we use the space of the W^2-valued distributions on $\mathbb{R}_t$, $\mathcal{D}'(\mathbb{R}_t;W^2)$, that is the set of the linear continuous maps from $C^\infty_0(\mathbb{R}_t)$ to W^2, we introduce the space W^{-1} defined as the dual space of W^1, endowed with its canonical norm. We warn that the elements of this space are not distributions but L^2 can be identified with a subspace of W^{-1}.

(1) The mixed problem for (3), (5), in $\mathcal{B}$, is well posed: Given $\Phi_0\in L^2$, $\Phi_1\in W^{-1}$, there exists a unique solution $\Phi\in\mathcal{D}'(\mathbb{R}_t;W^2)\cap C^0(\mathbb{R}_t;L^2)\cap C^1(\mathbb{R}_t;W^{-1})$ of (3) satisfying $\Phi(0,.)=\Phi_0(.)$, $\partial_t\Phi(0,.)=\Phi_1(.)$. Furthermore Φ satisfies for all $t\in\mathbb{R}$, $\|\partial_t\Phi(t)\|^2_{W^{-1}}+\|\Phi(t)\|^2_{L^2}=\|\Phi_1\|^2_{W^{-1}}+\|\Phi_0\|^2_{L^2}$. When $\Phi_0\in W^1$, $\Phi_1\in L^2$, then $\Phi\in\mathcal{D}'(\mathbb{R}_t;W^2)\cap C^0(\mathbb{R}_t;W^1)\cap C^1(\mathbb{R}_t;L^2)$ and $\|\partial_t\Phi(t)\|^2_{L^2}+\|\Phi(t)\|^2_{W^1}=\|\Phi_1\|^2_{L^2}+\|\Phi_0\|^2_{W^1}$.

(2) The solutions can be expanded into a discrete Kaluza–Klein tower: there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}\subset]0,\infty[$ with $\lim_{n\to\infty}\lambda_n=\infty$, and a Hilbert basis of $L^2(0,1)$, $(u_n)_{n\in\mathbb{N}}\subset H^1(]0,1[)\cap C^\infty(]0,1])$ with $u_n(0)=0$, $u_n'(1)+\frac{3}{2}u_n(1)=0$, such that for any $\Phi_0\in L^2$, $\Phi_1\in W^{-1}$, the solution Φ can be written as

$$\Phi(t,\mathbf{x},z)=\sum_0^\infty \phi_n(t,\mathbf{x})u_n(z) \tag{6}$$

where $\phi_n \in C^0(\mathbb{R}_t; L^2(\mathbb{R}^3_{\mathbf{x}})) \cap C^1(\mathbb{R}_t; H^{-1}(\mathbb{R}^3_{\mathbf{x}}))$ is solution of the Klein–Gordon equation $\partial_t^2\phi_n - \Delta_{\mathbf{x}}\phi_n + \lambda_n^2\phi_n = 0$, and the limit (6) holds in $C^0(\mathbb{R}_t; L^2) \cap C^1(\mathbb{R}_t; W^{-1})$. Moreover we have $\|\Phi_0\|^2_{L^2} + \|\Phi_1\|^2_{W^{-1}} = \sum_0^\infty \|\phi_n(t)\|^2_{L^2} + \|(-\Delta_{\mathbf{x}} + \lambda_n^2)^{-\frac{1}{2}}\partial_t\phi_n(t)\|^2_{L^2}$. When $\Phi_0 \in W^1$, $\Phi_1 \in L^2$, then $\phi_n \in C^0(\mathbb{R}_t; H^1(\mathbb{R}^3_{\mathbf{x}})) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}^3_{\mathbf{x}}))$, the limit (6) holds in $C^0(\mathbb{R}_t; W^1) \cap C^1(\mathbb{R}_t; L^2)$, and we have $\|\Phi_0\|^2_{W^1} + \|\Phi_1\|^2_{L^2} = \sum_0^\infty \|\nabla_{t,\mathbf{x}}\phi_n(t)\|^2_{L^2} + \lambda_n^2\|\phi_n(t)\|^2_{L^2}$.

The sequences λ_n and u_n are explicitly known in terms of Bessel functions. In particular, in the case of the gravitational fluctuations for which $\lambda = 2$, λ_n is the set of the strictly positive zeros of the Bessel function $J_1(x)$ and $u_n(z) = \sqrt{2z}J_2(\lambda_n z)/J_2(\lambda_n)$.

(3) We end this part by a result of uniform decay. We use a dyadic partition of the unity $\chi_p \in C_0^\infty([0,\infty[)$ satisfying $\sum_{p=0}^\infty \chi_p = 1$, $\operatorname{supp}\chi_0 \subset [0,2[$, $1 \leqslant p \Rightarrow \operatorname{supp}\chi_p \subset [2^{p-1}, 2^{p+1}]$. Given an integer $k \geqslant 0$, and two integer-valued functions k', k'' defined on $\mathbb{N}^3$, we introduce the functional space

$$\Theta\big(k,k',k''\big) := \Big\{\Phi \in L^2\big(\mathbb{R}^3_{\mathrm{x}}\times]0,1[_z\big);\ |\alpha| \leqslant k \quad\Rightarrow\quad \partial^\alpha_{\mathrm{x}}\Phi \in Dom\big(L^{\max(k'(\alpha),k''(\alpha))}\big),\ \|\Phi\|_{\Theta(k,k',k'')} < \infty\Big\},$$

where L is the operator $-\Delta_{\mathbf{x}} - \partial_z^2 + \frac{\lambda^2-\frac{1}{4}}{z^2}$ with the domain W^2. On this space we define the norm:

$$\|\Phi\|_{\Theta(k,k',k'')} := \sum_{|\alpha|\leqslant k}\left\|\big(1+|\mathrm{x}|\big)^{\frac{3}{2}}\partial^\alpha_{\mathrm{x}}\left(-\partial_z^2 + \frac{\lambda^2-\frac{1}{4}}{z^2}\right)^{k'(\alpha)}\Phi\right\|_{L^2(\mathbb{R}^3_{\mathrm{x}}\times]0,1[_z)}$$
$$+ \sum_{|\alpha|\leqslant k}\sum_{p=0}^{\infty}\left\|\chi_p\big(|\mathrm{x}|\big)\big(1+|\mathrm{x}|\big)^{\frac{3}{2}}\partial^\alpha_{\mathrm{x}}\left(-\partial_z^2 + \frac{\lambda^2-\frac{1}{4}}{z^2}\right)^{k''(\alpha)}\Phi\right\|_{L^2(\mathbb{R}^3_{\mathrm{x}}\times]0,1[_z)}.$$

We establish that there exists $C > 0$ such that for all $\Phi_0 \in \Theta(3,k',k_0'')$ and $\Phi_1 \in \Theta(2,k',k_1'')$, the solution Φ satisfies the following estimate:

$$\big|\Phi(t,\mathrm{x},z)\big| \leqslant C\big(|t|+|\mathrm{x}|\big)^{-\frac{3}{2}} z^{\lambda+\frac{1}{2}}\big(\|\Phi_0\|_{\Theta(3,k',k_0'')} + \|\Phi_1\|_{\Theta(2,k',k_1'')}\big),$$

with $k'(\alpha) = [\frac{\lambda+1}{2}] + 1$, $k_j''(\alpha) = [\frac{2\lambda+5-2|\alpha|-2j}{4}] + 1$.

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