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Algebra/Homological Algebra

Homological properties of noncommutative Iwasawa algebras

Propriétés homologiques des algèbres d'Iwasawa non commutatives

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A R T I C L E   I N F O

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A B S T R A C T

For any compact p-adic Lie group G, the Iwasawa algebra Ω_G is an Artin–Schelter Gorenstein algebra. We obtain the Auslander–Buchsbaum formula, the Bass's theorem and the No-holes theorem for noetherian modules over Λ_G and Ω_G, and the dual versions for their artinian modules. It is shown that Ω_G is Morita self-dual via dualizing complexes. We finally consider the homological invariant "grade" of filtered modules over Λ_G and Ω_G, when G is a uniform pro-p group with certain properties.

R É S U M É

Pour tout groupe de Lie G p-adique compact, l'algèbre d'Iwasawa Λ_G et son image épimorphique Ω_G sont des algèbres d'Artin–Schelter Gorenstein. Nous montrons la formule d'Auslander–Buchsbaum, le théorème de Bass et le théorème des « non trous » pour des modules noethériens sur Ω_G, ainsi que des versions duales pour leur modules artiniens. Il est montré que Ω_G est auto-duale au sens de Morita par des complexes dualisants. Finalement, nous considérons les invariants homologiques « *grade* » des modules filtrés sur Λ_G et Ω_G, lorsque G est un groupe uniforme pro-p satisfaisant certaines propriétés.

Version française abrégée

Soient G un groupe de Lie G p-adique compact et Λ_G (resp. Ω_G) son algèbre d'Iwasawa sur l'anneau $\mathbb{Z}_p$ des entiers p-adiques (resp. sur un corps fini $\mathbb{F}_p$ à p éléments). Des propriétés théoriques des anneaux et des propriétés homologiques des algèbres d'Iwasawa sont utiles pour comprendre la structure du dual de Pontryagin des groupes de Selmer et des autres modules sur les algèbres d'Iwasawa. Quelques travaux récents [2–4,15,16] sont dévoués aux propriétés théoriques des anneaux des algèbres d'Iwasawa. Certains aspects homologiques des algèbres d'Iwasawa ont été étudiés dans [2,6,13,15,16]. Par des méthodes homologiques et quelques techniques sur les algèbres non commutatives graduées, Venjakob a montré que si G est un pro-p groupe de Poincaré tel que Λ_G soit noethérien et que M soit un Λ_G-module à gauche finiment engendré, alors $\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} \Lambda_G$ [16]. C'est une version non commutative de la formule d'Auslander–Buchsbaum bien connue pour des modules finiment engendrés.

Le théorème de Venjakob amène deux questions : (a) est-ce que le Théorème 1.1 est vrai pour un groupe général G ? (b) est-ce que la formule de Bass et le théorème des « non trous » sont vrais pour Ω_G ? Dans cette Note, nous donnons d'abord des réponses affirmatives aux questions (a) et (b). Et puis, nous nous intéressons aux dualités de Morita de Ω_G

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par des complexes dualisants. Finalement, nous considérons les invariants homologiques «*grade*» des Λ_G-modules et Ω_G-modules finiment engendré par une bonne filtration, lorsque G est un groupe de Lie p-adique compact satisfaisant certaines propriétés.

1. Introduction

In recent years, there has been a resurgence in studying the noncommutative Iwasawa algebras of compact p-adic Lie groups. The Iwasawa theory for elliptic curves in arithmetic geometry is the main motivation to study the Iwasawa algebras Λ_G and Ω_G of a compact p-adic Lie group G (see [5,6,14,17,18]). Throughout, let p be a fixed prime integer and $\mathbb{Z}_p$ be the ring of p-adic integers. Let G be a compact p-adic Lie group. In this paper our objectives are the so-called *Iwasawa algebras* of G. They are the completed group algebras

$$\Lambda_G := \varprojlim \mathbb{Z}_p[G/N],$$

where the inverse limit is taken over the open normal subgroups N of G. Closely related to Λ_G is its epimorphic image Ω_G, which is defined as

$$\Omega_G := \varprojlim \mathbb{F}_p[G/N],$$

where $\mathbb{F}_p$ is the field of p elements. These algebras associated with certain topological setting were defined and studied by Lazard in his seminal 1965 paper [10] at first. They are complete noetherian semilocal algebras, which are in general noncommutative. Theorem C of [3] states that every prime ideal of the Iwasawa algebra Ω_G over any open torsion-free subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ is either zero or maximal. In this case, Ω_G is local and extremely noncommutative since the only non-zero prime ideal is the maximal ideal. Recall that an algebra A over a field is called *just infinite-dimensional* if it is infinite-dimensional and every non-zero ideal of A is finite codimensional. The Iwasawa algebras over $\mathrm{SL}_2(\mathbb{Z}_p)$ give rise to a class of just infinite-dimensional algebras. We refer to [2] and [10] for the basic properties of Λ_G and Ω_G.

Motivated by the main conjecture of Iwasawa theory, and more generally by the roles of Λ_G and Ω_G in the arithmetic geometry of elliptic curves and their Selmer groups, there has been considerable ring-theoretic activity concerning Λ_G and Ω_G (see [2–4,7,16–18]). Many Iwasawa algebras are nice examples of "just-infinite algebras" which both satisfy the Auslander–Gorenstein condition and are thus amenable to Lie theoretic analysis. Ring-theoretic and homological properties of the Iwasawa algebras are useful for understanding the structure of the Pontryagin dual of the Selmer groups [14,17] and other modules over the Iwasawa algebras [6,13]. Several recent papers [2–4,15,16] are devoted to ring-theoretic properties of the Iwasawa algebras. Certain homological aspects of the Iwasawa algebras have been studied in [2,6,13,15,16]. By homological methods and some techniques of noncommutative graded algebras Venjakob in [16] proved

Theorem 1.1 *(Venjakob's theorem). Let G be a pro-p Poincaré group such that Λ_G is noetherian. Let M be a finitely generated left Λ_G-module. Then*

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} \Lambda_G.$$

This is a noncommutative version of the well-known Auslander–Buchsbaum formula for finitely generated modules. Venjakob's theorem give rise to two questions: (a) does Theorem 1.1 hold for general G? (b) do the Bass formula and the No-holes theorem hold for Λ_G and Ω_G? We first give affirmative answers to the questions (a) and (b). Then we investigate the Morita dualities of Λ_G and Ω_G via dualizing complexes. We finally consider the homological invariant "grade" of finitely generated filtered Λ_G-modules and Ω_G-modules with a good filtration, when G is a compact p-adic Lie group with certain properties.

2. Preliminaries

Let us begin this section with some basic and important facts of Iwasawa algebras which we will use in the sequel. The following statement might be known, its interpretation is given for completeness. Note that it is actually concerned with Frobenius extensions.

Lemma 2.1. *Let $T = R * G$ be a crossed product of the ring R with the finite group G and let M_T be a right T-module. Then*

$$\operatorname{Ext}^i_T(M_T, T) \cong \operatorname{Ext}^i_R(M_R, R)$$

as left R-modules, for all $i \geqslant 0$.

As functors from the category of right T-modules to the category of left R-modules, $\operatorname{Ext}^i_R(-, R)$ and $\operatorname{Ext}^i_T(-, T)$ are derived from $\operatorname{Hom}_R(-, R_R)$ and from $\operatorname{Hom}_T(-, T_T)$, respectively. It is therefore sufficient to prove that

$$\mathrm{Hom}_T(M_T, T_T) \cong \mathrm{Hom}_R(M_R, R_R)$$

as left R-modules. Since $M_R \cong M_T \otimes_T T_R$, it is enough to show that $T \cong \mathrm{Hom}_R(T_R, R_R)$ as right T-modules. Let us define $\alpha : \mathrm{Hom}_R(T_R, R_R) \to T$ by $\alpha(f) = \sum_{g\in G} f(\bar{g})\bar{g}^{-1}$. One is easy to verify that α is a right T-module mapping. Since $\beta : T \to \mathrm{Hom}_R(T_R, R_R)$ given by $\beta(\sum_{g\in G} r_g\bar{g})(\bar{h}) = r_{h^{-1}}\overline{h^{-1}}\bar{h}$ is an inverse, the result follows.

The most important point of this result is that α is a well-defined (R, T)-bimodule homomorphism. If δ_x is the unique right R-module map $T \to R$ which sends $\bar{g}$ to 1 if $g = x$ and to 0 if $g \neq x$, then you can check that $\{\delta_x\colon x \in G\}$ is a basis for T as a left R-module and that $\alpha(\delta_x) = \bar{x}^{-1}$. Thus α is left R-linear and sends a basis for T as a left R-module to a basis for $\mathrm{Hom}_R(T_R, R_R)$ as a left R-module. This forces it to be an isomorphism. Using this, one can work out that the inverse β has to be of the form given above. In general crossed products, it is not true that $\bar{g}\overline{g^{-1}} = 1$. For example, if $R = \mathbb{Q}$ is the field of rational numbers and $T = \mathbb{Q}(\sqrt{2})$, then T is a crossed product of R with the cyclic group of order 2. However, in this case, it is impossible to for us to write $T = R \oplus R\bar{g}$ with $\bar{g}^2 = 1$.

Definition 2.2. Let A be a noetherian algebra over a fixed base field $\mathbb{K}$. We say A is *Artin–Schelter Gorenstein* if

(1) A has finite left and right injective dimension, say d,
(2) for every simple left A-module S, $\mathrm{Ext}^i_A(S, A) = 0$ for all $i \neq d$ and $\mathrm{Ext}^d(S, A)$ is a simple right A-module, and
(3) part (2) holds when 'left' and 'right' are exchanged.

Furthermore, A is said to be *Artin–Schelter regular* if A is Artin–Schelter Gorenstein and has global dimension d.

Recall that a ring R is *semilocal* if the factor of R by its Jacobson radical $J(R)$ is semisimple artinian. It is *local* if $R/J(R)$ is simple artinian, and *scalar local* if $R/J(R)$ is a division ring. It was well known that Ω_G and Λ_G are scalar local rings if and only if G is a pro-p group, which is due to Lazard [10]. Furthermore, by [2, Corollary 3.6] we know that if G is a uniform pro-p group, then Λ_G and Ω_G are noetherian, Auslander regular and scalar local domains. Similar to the connected graded case in [11], Ω_G is Artin–Schelter regular. In general case, for any compact p-adic Lie group G, there exists a normal open uniform subgroup H which is powerful pro-p of finite index. Applying the following facts

$$\Omega_G \cong \Omega_H * (G/H)$$

and Lemma 2.1 yields

$$\mathrm{Ext}^i_{\Omega_G}(M, \Omega_G) \cong \mathrm{Ext}^i_{\Omega_G}(M, \Omega_H).$$

This isomorphism, for any Ω_G-module M and for any $i \geqslant 0$, shows that Ω_G is AS Gorenstein. Thus we obtain

Proposition 2.3. *For any compact p-adic Lie group G, Ω_G is Artin–Schelter Gorenstein.*

Furthermore, we have a more general result.

Proposition 2.4. *Let B be an Artin–Schelter Gorenstein algebra and $\mathfrak{A} = B * G$ be a crossed product of B with the finite group G. Then $\mathfrak{A}$ is Artin–Schelter Gorenstein.*

It should be remarked that any simple A-module has finite length over B. The assertion follows from Lemma 2.1.

3. The main results

In this section we will study homological properties of the Iwasawa algebras $\Lambda = \Lambda_G$ and $\Omega = \Omega_G$ and those of finitely generated modules over them.

Let M be a finitely generated left Ω-module and let us denote the projective dimension and injective dimension of M by $\mathrm{pd}\, M$ and $\mathrm{id}\, M$, respectively. We define

$$\mathrm{depth}\, M = \min\bigl\{i \bigm| \mathrm{Ext}^i_\Omega(S, M) \neq 0 \text{ for some simple } \Omega\text{-module } S\bigr\}$$

and

$$\mathrm{codepth}\, M = \min\bigl\{i \bigm| \mathrm{Tor}^\Omega_j(S, M) \neq 0 \text{ for some simple } \Omega\text{-module } S\bigr\}.$$

Theorem 3.1. *Let G be a compact p-adic Lie group and $\Omega = \Omega_G$. Let M be a non-zero noetherian left Ω-module.*

(1) (*Auslander–Buchsbaum formula*) *If M has finite projective dimension, then*

$$\mathrm{pd}\, M + \mathrm{depth}\, M = \mathrm{depth}\, \Omega = \dim G.$$

(2) (*Bass theorem*) *If M has finite injective dimension, then*

$$\operatorname{id} M = \operatorname{id} \Omega = \dim G.$$

(3) (*No-holes theorem*) *For every integer i, there is a simple Ω-module S such that* $\operatorname{Ext}^i_\Omega(S, M) \neq 0$ *if and only if* $\operatorname{depth} M \leqslant i \leqslant \operatorname{id} M$.

This is due to Proposition 2.3 and [20, Theorem 0.1]. Furthermore, by the theory of Morita duality [21] we can get a dual version of Theorem 3.1.

Theorem 3.2. *Let G be a compact p-adic Lie group and $\Omega = \Omega_G$. Let M be a non-zero artinian left Ω-module.*

(1) *If M has finite injective dimension, then*

$$\operatorname{id} M + \operatorname{codepth} M = \operatorname{codepth} \Omega = \dim G.$$

(2) *If M has finite projective dimension, then*

$$\operatorname{pd} M = \operatorname{pd} \Omega = \dim G.$$

(3) *For every integer i, there is a simple Ω-module T such that* $\operatorname{Ext}^i_\Omega(M, T) \neq 0$ *if and only if* $\operatorname{depth} M \leqslant i \leqslant \operatorname{id} M$.

An algebra is called Quasi-Frobenius (or QF) if it is artinian and has injective dimension 0. The aforementioned Artin–Schelter Gorenstein algebras are actually common generalizations of QF algebras to higher injective dimension. It is well known that if an algebra A is QF, then the bimodule ${}_AA_A$ induces a Morita self-duality. Jategaonkar [9] showed that if A is a complete noetherian semilocal algebra such that $A/J(A)$ is finite-dimensional over its center, then A is Morita self-dual, where $J(A)$ is the Jacobson radical of A. But, not every complete noetherian semilocal algebra has a Morita duality. We naturally ask the following question: are the Iwasawa algebras $\Lambda = \Lambda_G$ and $\Omega = \Omega_G$ Morita self-dual? In particular, if G is a uniform pro-p group, then $\Omega_G/J(\Omega_G) \cong \mathbb{Z}_p$ by [2, Corollary 3.1]. In this case, Ω_G satisfies the condition of Jategaonkar's theorem and hence is Morita self-dual. By dualizing complexes, Morita dualities and the relation between them we will give a thorough answer to the previous question.

Let A be an algebra and A° be the opposite algebra of A. $D(A)$ ($D^b(A)$, $D^+(A)$ and $D^-(A)$, respectively) denotes the derived category of (bounded, left-bounded, right-bounded, respectively) complexes of A-modules. We refer to [8] for basic notions about complexes and derived categories. The noncommutative version of a dualizing complex was introduced by Yekutieli [22].

Definition 3.3. (See [22].) Let A be a left noetherian algebra and B be a right noetherian algebra. An object $R \in D^b(A \otimes B^\circ)$ is called a *dualizing complex over* (A, B) if it satisfies the following three conditions:

(1) R has finite injective dimension over A and B°.
(2) R has finite cohomology over A and B°.
(3) The canonical morphisms $B \to \mathrm{R}\operatorname{Hom}_A(R, R)$ and $A \to \mathrm{R}\operatorname{Hom}_{B^\circ}(R, R)$ are isomorphisms in $D(B \otimes B^\circ)$ and $D(A \otimes A^\circ)$, respectively.

When $A = B$, we say that R is a *dualizing complex over* A.

When we say that R is a dualizing complex over (A, B), we will assume implicitly that A is left noetherian and B is right noetherian. The next notion is a central object of this paper.

Definition 3.4. (See [22].) A dualizing complex R over (A, B) is *pre-balanced* if

(1) for every simple A-module S, $\operatorname{Ext}^i_A(S, R) = 0$ for all $i \neq 0$ and $\operatorname{Ext}^0_A(S, R)$ is a simple B°-module,
(2) the same statement holds when A and B° are exchanged.

Let R be a dualizing complex over (A, B) and let M be an A-module. The *grade* of M with respect to R is

$$j(M) = \inf\{q \mid \operatorname{Ext}^q_A(M, R) \neq 0\}.$$

The grade of a B°-module can be similarly defined. The *canonical dimension* with respect to a dualizing complex R is defined to be

$$\operatorname{Cdim} M = -j(M)$$

for all finitely generated A- (or B°-) modules M. A dualizing complex R over (A, B) is called Cdim-*symmetric* if for every (A, B)-bimodule which is finitely generated on both left and right sides, one has $\operatorname{Cdim}_A M = \operatorname{Cdim} M_B$.

Proposition 3.5. *Let $\mathbb{K}$ be a fixed base field, A be a left noetherian algebra over $\mathbb{K}$ and B be a right noetherian algebra over $\mathbb{K}$. Suppose that A and B are semilocal and complete with respect to their Jacobson radicals. Let R be a pre-balanced dualizing complex over (A, B). Then the following statements hold.*

(1) *There is a Morita duality between A and B induced by R.*
(2) *R is* Cdim*-symmetric.*

Theorem 3.6. *The Iwasawa algebra $\Omega = \Omega_G$ is Morita self-dual.*

Suppose that G is a compact p-adic Lie group of dimension d. Then Ω has finite injective dimension d and hence Ω is a dualizing complex over Ω. The Artin–Schelter Gorenstein condition (2, 3) shows that the complex shift $\Omega[d]$ is pre-balanced dualizing complex over Ω. This result follows from Proposition 3.5(1).

Proposition 3.7. *Let G be a compact p-adic Lie group of dimension d. For the Iwasawa algebra $\Omega = \Omega_G$, we have*

(1) *If $\mathfrak{p}$ is a minimal ideal of Ω, then* $\mathrm{Cdim}\,\Omega/\mathfrak{p} = d$.
(2) *Ω has a quasi-Frobenius artinian ring of fractions.*

Since Ω is an Artin–Schelter Gorenstein algebra of dimensions d, $R := \Omega[d]$ is a pre-balanced dualizing complex over Ω. By Proposition 3.5(2), R is Cdim_R-symmetric. Shifting $R = \Omega[d]$ back to Ω, the Cdim changes by $+d$. So Ω is Cdim_Ω-symmetric. Let the dimension function δ in [1, Theorem 6.1] be Cdim_Ω. On the other hand, we know that Ω is Auslander–Gorenstein by [2, Theorem 5.2]. The assertions follow from [1, Theorem 6.1].

We will end the paper with another important homological invariant "grade". Suppose that A is a noetherian algebra and that M is a finitely generated left A-module. The *grade number* of M, which is denoted by $\mathrm{grade}_A M$, is the unique smallest integer k such that $\mathrm{Ext}_A^k(M, A) \neq 0$. In view of [12, Chapter III, Theorem 2.2.5], if A is a Zariskian filtered ring, then for any filtered A-module M with a good filtration $\{F_iM \mid i \in \mathbb{Z}\}$ we have $\mathrm{grade}_A M \geqslant \mathrm{grade}_{\mathrm{gr}\,A} \mathrm{gr}\,M$. Furthermore, for any noetherian filtered algebra A and any finitely generated filtered A-module M, its nicest feature is that the homological identity $\mathrm{grade}_A M = \mathrm{grade}_{\mathrm{gr}\,A} \mathrm{gr}\,M$ is true when M is a good filtered A-module M and $\mathrm{gr}\,A$ is regular [12, Chapter III, Theorem 2.5]. We prove this equation under "module-wise" conditions and then apply this equation to a wide classes of filtered rings, such as the Iwasawa algebras. Let A be a filtered algebra with a filtration $\{F_iA \mid i \in \mathbb{N}\}$ and $f_i : F_iA \to F_iA/F_{i-1}A$ be a natural homomorphism. We set

$$\mathrm{gr}\,A = \bigoplus_{i=0}^{\infty} F_iA/F_{i-1}A \quad (F_{-1}A = 0).$$

Then $\mathrm{gr}\,A$ is a graded ring with multiplication

$$f_i(a)f_j(b) = f_{i+j}(ab), \quad a \in F_iA,\ b \in F_jA.$$

Let M be a filtered A-module with a filtration $\{F_iM \mid i \in \mathbb{Z}\}$ and $g_i : F_iM \to F_iM/F_{i-1}M$ be a natural homomorphism. Let us set

$$\mathrm{gr}\,M = \bigoplus_{i\in\mathbb{Z}} F_iM/F_{i-1}M.$$

Then $\mathrm{gr}\,M$ is a graded $\mathrm{gr}\,A$-module by

$$f_i(a)g_j(m) = g_{i+j}(am), \quad a \in F_iA,\ m \in F_jM.$$

For a filtered A-module M with the filtration $\{F_iM \mid i \in \mathbb{Z}\}$, its filtration is said to be *good* if there exist $i_k \in \mathbb{Z}$ and $m_k \in M$ $(1 \leqslant k \leqslant r)$ such that

$$F_iM = \sum_{k=1}^{r} (F_{i-i_k}A)m_k$$

for all $i \in \mathbb{Z}$. By [12, Chapter I, 5.2] we know that the following three conditions are equivalent:

(1) M has a good filtration;
(2) $\mathrm{gr}\,M$ is a finitely generated $\mathrm{gr}\,A$-module for a filtration $\{F_iM \mid i \in \mathbb{Z}\}$;
(3) M is a finitely generated A-module.

Theorem 3.8. *Let R be a filtered ring such that* $\mathrm{gr}\,R$ *is a commutative noetherian ring and M is a filtered R-module with a good filtration. If* $\mathrm{gr}\,M$ *has finite Gorenstein dimension, then the equality* $\mathrm{grade}_R M = \mathrm{grade}_{\mathrm{gr}\,R} \mathrm{gr}\,M$ *holds.*

Let G be a uniform pro-p group of dimension d. Then the Iwasawa algebra Ω_G of G is a local ring with maximal ideal $\mathfrak{M}$. The $\mathfrak{M}$-adic filtration on Ω_G is defined as follows: $F_i\Omega_G = \mathfrak{M}^{-i}$ for all $i \leqslant 0$ and $F_i\Omega_G = \Omega_G$ for $i \geqslant 0$. By [2, Theorem 3.4] we know that the graded ring $\mathrm{gr}_{\mathfrak{M}}\,\Omega_G$ of Ω_G with respect to the $\mathfrak{M}$-adic filtration is isomorphic to a polynomial ring in $d = \dim G$ variables:

$$\mathrm{gr}_{\mathfrak{M}}\,\Omega_G \cong \mathbb{F}_p[X_1, \ldots, X_d].$$

If G is a uniform and extra-powerful pro-p group of dimension d, then the Iwasawa algebra Λ_G of G is a local ring with maximal ideal $\mathfrak{M}$. By [16, Theorem 3.22 and Lemma 3.25] one can induce an $\mathfrak{M}$-adic filtration on Λ_G. Furthermore, the induced $\mathfrak{M}$-adic filtration leads to the following isomorphic relation

$$\mathrm{gr}_{\mathfrak{M}}\,\Lambda_G \cong \mathbb{F}_p[X_1, \ldots, X_{d+1}].$$

Therefore, $\mathrm{gr}\,\Omega_G$ is a commutative Gorenstein algebra when G is a uniform pro-p group and $\mathrm{gr}\,\Lambda_G$ is a commutative Gorenstein algebra when G is a uniform and extra-powerful pro-p group.

Corollary 3.9. *Let G be a compact p-adic Lie group such that $\mathrm{gr}\,\Lambda_G$ (resp. $\mathrm{gr}\,\Omega_G$) are commutative Gorenstein algebras. Suppose that M is a finitely generated filtered Λ_G-module (resp. Ω_G-module) with a good filtration. Then the equality $\mathrm{grade}_{\Lambda_G} M = \mathrm{grade}_{\mathrm{gr}\,\Lambda_G} \mathrm{gr}\,M$ (resp. $\mathrm{grade}_{\Omega_G} M = \mathrm{grade}_{\mathrm{gr}\,\Omega_G} \mathrm{gr}\,M$) holds.*

4. Conclusion

As can be seen, one sharp distinction between the Iwasawa algebra Λ_G and the Iwasawa algebra Ω_G is that Ω_G is an algebra over the field $\mathbb{F}_p$, but Λ_G is an algebra over the commutative ring $\mathbb{Z}_p$. Most part of the current article mainly treat homological properties of Ω_G and those of finitely generated modules over Ω_G. In a forthcoming article [19] we will obtain similar results for Λ_G instead of Ω_G. The case of Λ_G is much more difficult than that of Ω_G.

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