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Partial Differential Equations

On a new class of functions related to *VMO*

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ABSTRACT

In this Note, we compare the space *VMO* and the spaces

$$\mathbf{I}_p := \left\{ g \in L^1(\Omega;\mathbb{R});\ \int\limits_{\Omega}\int\limits_{\Omega}_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{d+p}}\,\mathrm{d}x\,\mathrm{d}y < +\infty \quad \forall\delta > 0 \right\}$$

where Ω is a bounded open subset of $\mathbb{R}^d$, $d \geqslant 1$, and $p \geqslant 0$. In particular, we prove that $\mathbf{I}_d \subset VMO$. This sharpens the well-known result stating that $W^{s,p} \subset VMO$ for $0 < s < 1$ and $sp = d$. Moreover, we establish that *VMO* is much bigger than $\mathbf{I}_d$ by showing that $VMO \not\subset \mathbf{I}_1$. We also present some results when the double integral above is taken on the set $\{(x, y) \in \Omega \times \Omega;\ |e^{ig(x)} - e^{ig(y)}| > \delta\}$.

RÉSUMÉ

Dans cette Note, nous comparons l'espace *VMO* et les espaces

$$\mathbf{I}_p := \left\{ g \in L^1(\Omega;\mathbb{R});\ \int\limits_{\Omega}\int\limits_{\Omega}_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{d+p}}\,\mathrm{d}x\,\mathrm{d}y < +\infty \quad \forall\delta > 0 \right\},$$

où Ω est un ouvert borné de $\mathbb{R}^d$, $d \geqslant 1$, et $p \geqslant 0$. En particulier, nous prouvons que $\mathbf{I}_d \subset VMO$. Ceci améliore le résultat bien connu affirmant que $W^{s,p} \subset VMO$ pour $0 < s < 1$ et $sp = d$. D'autre part, nous prouvons que *VMO* est plus grand que $\mathbf{I}_d$; en fait $VMO \not\subset \mathbf{I}_1$. Nous présentons aussi des résultats lorsque l'intégrale double ci-dessus est prise sur l'ensemble $\{(x, y) \in \Omega \times \Omega;\ |e^{ig(x)} - e^{ig(y)}| > \delta\}$.

1. Main results

The principal motivation of this note comes from the study of the topological degree of maps from the sphere $\mathbb{S}^d$ into itself. It was proved in [2] that the degree is well-defined for maps $u \in VMO(\mathbb{S}^d, \mathbb{S}^d)$. In fact it suffices to assume that

$$\limsup_{|Q|\to 0} \fint_Q \left| u(x) - \fint_Q u(y)\,\mathrm{d}y \right| \mathrm{d}x < 1; \tag{1.1}$$

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and the constant 1 is optimal. Note that (1.1) is satisfied in particular if

$$\limsup_{|Q|\to 0} \fint_Q \fint_Q \big|u(x)-u(y)\big|\,dx\,dy < 1/2.$$

On the other hand, it was proved in [6] that the degree of u is well-defined when

$$\iint\limits_{\substack{\mathbb{S}^d\,\mathbb{S}^d\\ |u(x)-u(y)|>\delta}} \frac{1}{|x-y|^{2d}}\,dx\,dy < +\infty \quad \text{for some } \delta \in (0,\ell_d), \tag{1.2}$$

where $\ell_d = \sqrt{2+2/(d+1)}$; and moreover

$$|\deg u| \leqslant C_d \iint\limits_{\substack{\mathbb{S}^d\,\mathbb{S}^d\\ |u(x)-u(y)|\geqslant \ell_d}} \frac{1}{|x-y|^{2d}}\,dx\,dy. \tag{1.3}$$

Therefore it is natural to investigate the possible connection between the spaces *VMO*, *BMO*, and the class of functions satisfying conditions of the type (1.2). We introduce the following definitions. Let Ω be a smooth bounded domain in $\mathbb{R}^d$, and $0 \leqslant p < +\infty$. Set

$$\mathbf{I}_p = \Bigg\{ g \in L^1(\Omega;\mathbb{R});\ \iint\limits_{\substack{\Omega\,\Omega\\ |g(x)-g(y)|>\delta}} \frac{1}{|x-y|^{d+p}}\,dx\,dy < +\infty\ \forall \delta > 0 \Bigg\}$$

and

$$\mathbf{J}_p = \Bigg\{ g \in L^1(\Omega;\mathbb{R});\ \iint\limits_{\substack{\Omega\,\Omega\\ |g(x)-g(y)|>\delta}} \frac{1}{|x-y|^{d+p}}\,dx\,dy < +\infty \text{ for some } \delta > 0 \Bigg\}.$$

The case $p<0$ is not interesting because $\mathbf{I}_p = \mathbf{J}_p$ coincides with $L^1(\Omega)$.

Here is a brief list of properties:

A) $\mathbf{I}_p$ and $\mathbf{J}_p$ are vector spaces.
B) $\mathbf{I}_p \subset \mathbf{I}_q$ and $\mathbf{J}_p \subset \mathbf{J}_q$ if $p \geqslant q$.
C) $C(\bar{\Omega}) \subset \mathbf{I}_p \subset \mathbf{J}_p$ for all $p \geqslant 0$.
D) $W^{s,p} \subset \mathbf{I}_{sp}$ for all $s \in (0,1)$ and $p>1$.
We recall here that, for $0<s<1$ and $p>1$,

$$W^{s,p}(\Omega) := \big\{ g \in L^p(\Omega);\ |g|_{W^{s,p}} < +\infty \big\},$$

where

$$|g|^p_{W^{s,p}} := \iint\limits_{\Omega\,\Omega} \frac{|g(x)-g(y)|^p}{|x-y|^{d+sp}}\,dx\,dy.$$

E) $W^{1,p} \subset \mathbf{I}_p$ for all $p>1$. More precisely (see [5]), for $p>1$ and $g \in W^{1,p}(\Omega)$, we have

$$\delta^p \iint\limits_{\substack{\Omega\,\Omega\\ |g(x)-g(y)|>\delta}} \frac{1}{|x-y|^{d+p}}\,dx\,dy \leqslant C_{d,p,\Omega} \int\limits_{\Omega} |\nabla g|^p\,dx.$$

The constant $C_{d,p,\Omega}$ blows up as $p \to 1$ and in fact $W^{1,1} \not\subset \mathbf{I}_1$ (an example due to A. Ponce is presented in [5]).
F) $\mathbf{J}_p \subset L^{p^*}$ with $1<p<d$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$ (see [7]). This is an extension of the classical Sobolev embedding $W^{1,p} \subset L^{p^*}$. It is not true that $\mathbf{I}_d \subset L^\infty$ (clearly $W^{s,p} \subset \mathbf{I}_d$ and it is known that $W^{s,p} \not\subset L^\infty$ for $sp=d$). Even when $p>d$, it is not true that $\mathbf{I}_p \subset L^\infty$ (see [7]); this is in contrast with the Morrey–Sobolev embedding.

It is known that $W^{s,p} \subset VMO$ for all $d \geqslant 1$, $0<s\leqslant 1$, and $sp=d$; see e.g. [2]. In view of D), one may wonder whether the larger space $\mathbf{I}_d$ is also contained in *VMO*. The answer is positive:

Theorem 1. *Let $d \geqslant 1$. Then*

a) $\mathbf{J}_d \subset BMO$.
b) $\mathbf{I}_d \subset VMO$.

Remark 1. The exponent d in Theorem 1 is optimal in the following sense: if $d \geqslant 1$ and $0 \leqslant p < d$ then $\mathbf{I}_p \not\subset BMO$. Indeed, let $q > 1$ and $0 < s < 1$ be such that $p < sq < d$. Then

$$W^{s,q} \subset \mathbf{I}_{sq} \subset \mathbf{I}_p \quad \text{and} \quad W^{s,q} \not\subset BMO.$$

This implies $\mathbf{I}_p \not\subset BMO$.

The proof of Theorem 1 is essentially based on the following proposition which is proved in [7]. In what follows, we denote by Q the unit cube in $\mathbb{R}^d$.

Proposition 1. *Let $d \geqslant 1$, $p \geqslant 1$, $\delta > 0$, and $g \in L^1(Q)$. Then*

$$\int_Q\int_Q \left|g(x) - g(y)\right|^p \,\mathrm{d}x\,\mathrm{d}y \leqslant C_{d,p}\Bigg[\mathop{\int_Q\int_Q}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{d+p}}\,\mathrm{d}x\,\mathrm{d}y + \delta^p\Bigg], \tag{1.4}$$

for some positive constant $C_{d,p}$ depending only on d and p.

Remark 2. The proof of Proposition 1 is quite delicate and it would be desirable to find a more elementary argument, even for $d = 1$. It makes use of ideas introduced in Bourgain–Nguyen [1]. It also relies on the John–Nirenberg inequality [4]. Some inequalities related to (1.4) and their applications in the theory of Sobolev spaces are presented in [7].

One may ask whether the inclusions in Theorem 1 are strict. It turns out that *VMO* is "much bigger" than $\mathbf{I}_d$. In fact, we have a stronger assertion:

Theorem 2. *Let $d \geqslant 1$. Then there exists $g \in VMO$ such that $g \in W^{s,p}$ for all $s \in (0,1)$, $p > 1$ with $sp < 1$, and $g \notin \mathbf{J}_1$, i.e.,*

$$\mathop{\int_Q\int_Q}_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{d+1}}\,\mathrm{d}x\,\mathrm{d}y = +\infty, \quad \forall \delta > 0.$$

Remark 3. Let $0 \leqslant t < 1$ and $d \geqslant 1$. We have not been able to construct a function $g \in VMO$ such that $g \notin \mathbf{J}_t$. It might be true, for example, that $VMO \subset \mathbf{J}_0$; this is an open problem.

We next present a variant of Proposition 1.

Theorem 3. *Let $1 \leqslant p < +\infty$ and $0 < \delta < \sqrt{3}$. We have, for all $g \in C(\bar{Q}, \mathbb{R})$.*

$$\int_Q\int_Q \left|g(x) - g(y)\right|^p \,\mathrm{d}x\,\mathrm{d}y \leqslant C_{d,p,\delta}\Bigg(\mathop{\int_Q\int_Q}_{|e^{ig(x)}-e^{ig(y)}|>\delta} \frac{1}{|x-y|^{d+p}}\,\mathrm{d}x\,\mathrm{d}y + 1\Bigg). \tag{1.5}$$

Moreover, the restriction that $\delta < \sqrt{3}$ is optimal.

Theorem 3 has been proved in [3] when $p = 1$ and $d = 1$. Already in this case the proof is quite elaborate. The case $d = 1$ and $p > 1$ can be proved using exactly the same argument as in the case $d = 1$ and $p = 1$. The proof in the case $d > 1$ is a consequence of the 1-d case using the argument in Step 2 of the proof of [7, Theorem 1].

Theorem 3 fails if we delete the assumption that $g \in C(\bar{Q})$. In fact, for each $n \in \mathbb{N}_+$, take $g_n(x) = 0$ on $(0, 1/2) \times (0,1)^{N-1}$ and $g_n(x) = 2\pi n$ for $x \in (1/2, 1) \times (0,1)^{N-1}$. Then

$$\int_Q\int_Q \left|g_n(x) - g_n(y)\right|^p \,\mathrm{d}x\,\mathrm{d}y \to \infty \quad \text{as } n \to \infty,$$

and

$$\int\limits_{Q}\int\limits_{Q}_{|e^{ig_n(x)}-e^{ig_n(y)}|>\delta} \frac{1}{|x-y|^{d+p}}\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall \delta > 0.$$

Theorem 3 implies Proposition 1 when $g \in C(\bar{Q})$. However we do not know how to deduce Proposition 1 from Theorem 3 for a general function $g \in L^1(Q)$ because we are not able to pass to the limit in the RHS of (1.4) when g is regularized.

Another natural question is whether (1.5) holds for $g \in VMO(Q)$. We know that the answer is positive if $d = 1$ and $p = 1$ (see [3]). By the same method as in [3], one can prove that the answer holds for $d = 1$ and $p > 1$.

We also have

Theorem 4. *Let $d \geqslant 1$ and $k \in \mathbb{N}_+$ be such that $1 \leqslant k \leqslant d$. Then there exists $g \in VMO(Q)$ such that $g \in W^{s,p}(Q)$ for all $s \in (0,1)$, $p > 1$, and $sp < k$, and*

$$\int\limits_{Q}\int\limits_{Q}_{|e^{ig(x)}-e^{ig(y)}|>\delta} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{d+k}} = +\infty, \quad \forall\, 0 < \delta < 2. \tag{1.6}$$

Detailed proofs of these results will be presented elsewhere.

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