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Lie Algebras/Functional Analysis

Biquantization techniques for computing characters of differential operators on Lie groups

Techniques de bi-quantification pour le calcul des caractères des opérateurs différentiels sur les groupes de Lie

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A B S T R A C T

We compare the character of the algebra $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$, as used by Fujiwara and Corwin and Greenleaf, with the character produced from biquantization techniques applied in the Lie case by Cattaneo and Torossian. We prove that up to a smaller (specialization) algebra, these two characters are the same. An old example is also treated and it is proved that we now get more information about the question of when the symmetrization is an isomorphism of algebras.

R É S U M É

Nous comparons le caractère de l'algèbre $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$, tel qu'utilisé par Fujiwara et Corwin–Greenleaf, avec le caractère produit par les techniques de bi-quantification appliquées au cas des algèbres de Lie par Cattaneo–Torossian. Nous démontrons que ces deux caractères coïncident, à une algèbre (de spécialisation) plus petite près. Nous discutons également un exemple bien connu et nous obtenons des informations supplémentaires quant à la question de savoir si la symétrisation est un isomorphisme d'algèbres.

1. Introduction

Let G be a real nilpotent, connected and simply connected Lie group with $\mathfrak{g}$ its Lie algebra, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$, $\lambda \in \mathfrak{h}^*$ such that $\lambda([\mathfrak{h},\mathfrak{h}]) = 0$. Then for $Y \in \mathfrak{h}$, $\chi_\lambda : H \to \mathbb{C}$ defined by $\chi_\lambda(\exp Y) = e^{i\lambda(Y)}$, is a unitary character of H and we define the induced representation $\tau_\lambda := Ind(G \uparrow H, \chi_\lambda)$ with Hilbert space $\mathcal{H}_\lambda := L^2(G, H, \lambda)$ the separable completion of $C_c^\infty(G, H, \chi_\lambda)$ with respect to the norm $\|\phi\|_2 = \int_{G/H} |\phi(g)|^2 \, \mathrm{d}_{G/H}(g)$. The action of G on $\phi \in L^2(G, H, \lambda)$ is translation on the left: $\tau_\lambda(g)(\phi)(g') = \phi(g^{-1}g')$. These data correspond to a line bundle $\mathcal{L}_\lambda$ with base space G/H and space of sections these functions ϕ. Let $\mathcal{H}_\lambda^{-\infty}$ be the space of antilinear continuous forms on $\mathcal{H}_\lambda^\infty$, the later being the space of C^∞-vectors of $\mathcal{H}_\lambda$. The action of $U_{\mathbb{C}}(\mathfrak{g})$ on $\mathcal{H}_\lambda^{-\infty}$ will be denoted by $\mathrm{d}\tau_\lambda^{-\infty}$ and the action of $U_{\mathbb{C}}(\mathfrak{g})$ on $\mathcal{H}_\lambda^\infty$ is denoted respectively by $\mathrm{d}\tau_\lambda^\infty$.

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Penney vectors. Let $f \in \mathfrak{g}^*$ such that $f|_{\mathfrak{h}} = \lambda$ and $\mathfrak{b}$ a polarization with respect to f. We denote B its associated Lie group. Set $\mathfrak{h}_{if} := \langle H + if(H), H \in \mathfrak{h}\rangle$. We shall denote by $U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{-if}$ the ideal of $U_{\mathbb{C}}(\mathfrak{g})$ generated by $\mathfrak{h}_{-if}$. Let also $\mathrm{d}_{H,H\cap B}$ be a left-invariant measure on $H/H\cap B$. Let α_f be an element of $\mathcal{H}_f^{-\infty}$ defined for $\phi \in \mathcal{H}_f^{\infty}$, as

$$\langle \alpha_f, \phi\rangle = \int\limits_{H/H\cap B} \overline{\phi(h)\chi_\lambda(h)}\, \mathrm{d}_{H/H\cap B}(h). \tag{1}$$

The vector α_f is H-semi-invariant [3]. Because of this invariance property of α_f, the algebra $(U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{-if})^{\mathfrak{h}}$ is acting on α_f.

In an algebraic setting the lagrangian condition can be rewritten as

$$\exists \mathcal{O} \subset \lambda + \mathfrak{h}^{\perp} \text{ a non-empty Zariski-open set, such that } \forall f \in \mathcal{O},\ \dim(\mathfrak{h}\cdot f) = \frac{1}{2}\dim(\mathfrak{g}\cdot f). \tag{2}$$

Recall that if the $H \cdot f$ orbits are lagrangian in the $G \cdot f$ orbits, then $(U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{-if})^{\mathfrak{h}}$ is commutative (for this result see [7, §5, Theorem 5.4 and Corollary 5.5]).

2. Construction of characters

One of our main objects is the reduction algebra $H^0_{(\epsilon)}(\mathfrak{h}_\lambda^{\perp}, d^{(\epsilon)}_{\mathfrak{h}_\lambda^{\perp},\mathfrak{q}})$. To briefly describe it we first need to describe the differential $d^{(\epsilon)}_{\mathfrak{h}_\lambda^{\perp},\mathfrak{q}} : S(\mathfrak{q})[\epsilon] \to S(\mathfrak{q})[\epsilon] \otimes \mathfrak{h}^*$. This differential is defined as $d^{(\epsilon)}_{\mathfrak{h}_\lambda^{\perp},\mathfrak{q}} := \sum_{i=1}^{\infty} \epsilon^i d^{(i)}_{\mathfrak{h}_\lambda^{\perp},\mathfrak{q}}$ where $d^{(i)}_{\mathfrak{h}_\lambda^{\perp},\mathfrak{q}} := \sum_{\Gamma\in\mathcal{B}_i\cup\mathcal{BW}_i} \overline{\omega}_\Gamma B_\Gamma$. Here Γ stands for Kontsevich graphs (as in [10]) that have to belong to $\mathcal{B}_i \cup \mathcal{BW}_i$, a special family of Kontsevich graphs (namely Bernoulli and Bernoulli attached to a wheel, see [6] for their description). The component $\overline{\omega}_\Gamma$ is a real coefficient depending on Γ and B_Γ is a differential operator depending also on Γ. For more details on the definitions and the formulas we refer to [1, §2.3.2, §3.2.1], or the note [2]. The elements of this algebra are polynomials (on the formal deformation parameter ϵ) $P_{(\epsilon)}$, which are solutions of the equation $d^{(\epsilon)}_{\mathfrak{h}_\lambda^{\perp},\mathfrak{q}}(P_{(\epsilon)}) = 0$. The space of solutions is a vector space which we equip with the Cattaneo–Felder (associative) star-product $*_{CF,\epsilon}$ to take the reduction algebra $H^0_{(\epsilon)}(\mathfrak{h}_\lambda^{\perp}, d^{(\epsilon)}_{\mathfrak{h}_\lambda^{\perp},\mathfrak{q}})$.

Choose a supplementary space $\mathfrak{q}$ for $\mathfrak{h}$ in $\mathfrak{g}$. Let $Y \in \mathfrak{g}$, set $q(Y) := \det_{\mathfrak{g}}(\frac{\sinh\frac{\mathrm{ad}Y}{2}}{\frac{\mathrm{ad}Y}{2}})$, and recall the symmetrization map $\beta : S(\mathfrak{g}) \to U(\mathfrak{g})$. Define T_1, T_2 to be the operators

$$T_1 : H^0_{(\epsilon)}\big(\mathfrak{g}^*, d^{(\epsilon)}_{\mathfrak{g}^*}\big) \to H^0_{(\epsilon)}\big(\mathfrak{h}_\lambda^{\perp}, d^{(\epsilon)}_{\mathfrak{g}^*,\mathfrak{h}_\lambda^{\perp},\mathfrak{q}}\big), \qquad F \mapsto F *_1 1,$$
$$T_2 : H^0_{(\epsilon)}\big(\mathfrak{h}_\lambda^{\perp}, d^{(\epsilon)}_{\mathfrak{h}_\lambda^{\perp},\mathfrak{q}}\big) \to H^0_{(\epsilon)}\big(\mathfrak{h}_\lambda^{\perp}, d^{(\epsilon)}_{\mathfrak{g}^*,\mathfrak{h}_\lambda^{\perp},\mathfrak{q}}\big), \qquad G \mapsto 1 *_2 G,$$

that is the operators defining the Cattaneo–Felder bimodule structure on the biquantization diagram of $\mathfrak{g}^*$ and $\mathfrak{h}_\lambda^{\perp}$. We denoted as $H^0_{(\epsilon)}(\mathfrak{h}_\lambda^{\perp}, d^{(\epsilon)}_{\mathfrak{g}^*,\mathfrak{h}_\lambda^{\perp},\mathfrak{q}})$ the reduction space at the corner of this diagramm and as $*_1, *_2$ its left $H^0_{(\epsilon)}(\mathfrak{g}^*, d^{(\epsilon)}_{\mathfrak{g}^*})$-module structure and its right $H^0_{(\epsilon)}(\mathfrak{h}_\lambda^{\perp}, d^{(\epsilon)}_{\mathfrak{h}_\lambda^{\perp},\mathfrak{q}})$-module structure, respectively. Using some simple facts we write them as $T_1 : (S_{(\epsilon)}(\mathfrak{g}), *_{DK}) \simeq (U_{(\epsilon)}(\mathfrak{g}), \cdot) \to S(\mathfrak{q})[\epsilon]$, $T_2 : H^0_{(\epsilon)}(\mathfrak{h}_\lambda^{\perp}, d^{(\epsilon)}_{\mathfrak{h}_\lambda^{\perp},\mathfrak{q}}) \to S(\mathfrak{q})[\epsilon]$.

The PBW theorem holds for the deformed algebras $S_{(\epsilon)}(\mathfrak{g}) = S(\mathfrak{q})[\epsilon] \oplus S_{(\epsilon)}(\mathfrak{g}) *_{DK} \mathfrak{h}_\lambda$ ($*_{DK}$ stands for the Duflo–Kontsevich star-product) and $U_{(\epsilon)}(\mathfrak{g})$ and there is a symmetrization map $\beta_{(\epsilon)} : S_{(\epsilon)}(\mathfrak{g}) \to U_{(\epsilon)}(\mathfrak{g})$. We denote $\overline{\beta}_{\mathfrak{q},(\epsilon)} : S(\mathfrak{q})[\epsilon] \to U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda$ the quotient of this symmetrization map with respect to the chosen $\mathfrak{q}$. We now write $\forall X \in \mathfrak{g}, q_{(\epsilon)}(X) := q(\epsilon X)$ and note that using the isomorphism $(S_{(\epsilon)}(\mathfrak{g}), *_{DK}) \simeq (S_{(\epsilon)}(\mathfrak{g}), *_{CF}) \simeq (U_{(\epsilon)}(\mathfrak{g}), \cdot)$, $U_{(\epsilon)}(\mathfrak{g})$ can be decomposed as $U_{(\epsilon)}(\mathfrak{g}) = \bar{\beta}_{\mathfrak{q},(\epsilon)} \circ \partial_{q^{1/2}_{(\epsilon)}}(S(\mathfrak{q})[\epsilon]) \oplus U_{(\epsilon)}(\mathfrak{g}) \cdot \mathfrak{h}_\lambda$. Finally we will write $\overline{T}_1 := T_1|_{S(\mathfrak{q})[\epsilon]}$. It takes a small lemma to show that $\overline{T}_1$ is an isomorphism of vector spaces and we will denote abusively by $\overline{T}_1^{-1}$ its inverse $\overline{T}_1^{-1} : S(\mathfrak{q})[\epsilon] \to S(\mathfrak{q})[\epsilon] \subset H^0_{(\epsilon)}(\mathfrak{g}^*, d^{(\epsilon)}_{\mathfrak{g}^*})$.

In [1, §3.4.2, Theorem 3.1] and the note [2] we proved that there is an explicit non-canonical isomorphism

$$\overline{\beta}_{\mathfrak{q},(\epsilon)} \circ \partial_{q^{1/2}_{(\epsilon)}} \circ \overline{T}_1^{-1} T_2 : H^0_{(\epsilon)}\big(\mathfrak{h}_\lambda^{\perp}, d^{(\epsilon)}_{\mathfrak{h}_\lambda^{\perp},\mathfrak{q}}\big) \xrightarrow{\simeq} \big(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda\big)^{\mathfrak{h}}.$$

We shall use this fact to construct a family of characters by means of Cattaneo–Felder–Torossian techniques [4,5]. More specifically,

Theorem 1. *(See [6].) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$, $\mathfrak{h} \subset \mathfrak{g}$, $f \in \mathfrak{g}^*$ such that $\mathfrak{h}$ is lagrangian with respect to f. Let $\mathfrak{b}$ be a polarization of f and $\mathfrak{q}_{\mathfrak{b}}$ a transverse supplementary of $\mathfrak{h}$. The map*

$$\gamma_{CT} : \left(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_f\right)^{\mathfrak{h}} \to \mathbb{R}[\epsilon] \qquad u \mapsto \bar{T}_1^L \circ \bar{\beta}^{-1}_{\mathfrak{q}_{\mathfrak{b}},(\epsilon)}(u)(f)$$

is a character of $(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$.

Modifying the initial conditions we actually get something more useful: The previous theorem constructs only one character. However if the Lie group G is nilpotent, under the general lagrangian condition ($\exists \mathcal{O} \subset -\lambda + \mathfrak{h}^\perp$ such that $\forall l \in \mathcal{O}$, the orbits $H \cdot l \subset G \cdot l$ are lagrangian submanifolds), we can construct a character of $(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$ for each such element l. Our goal here is to compare the character defined through the Penney eigendistribution in non-commutative harmonic analysis with that of deformation quantization.

Let $\mathcal{H}_f^\infty$ be the C^∞-vectors of the Hilbert space of the representation $\tau_f = Ind(G, H, f)$ and $\alpha_f \in \mathcal{H}_f^{-\infty}$ the distribution defined for $\phi \in \mathcal{H}_f^\infty$ from the formula $\langle \alpha_f, \phi \rangle = \int_{H/H\cap B} \overline{\phi(h)\chi_\lambda(h)}\, \mathrm{d}_{H/H\cap B}(h)$.

Theorem 2. *(See [8].) Let* $\mathfrak{g}$ *be a Lie algebra* $(\dim(\mathfrak{g}) < \infty)$, $\mathfrak{h} \subset \mathfrak{g}$, λ *a character of* $\mathfrak{h}$. *Suppose that generically the representation* $\tau_\lambda = Ind(G, H, \lambda)$ *has finite multiplicities in her spectral decomposition. Then for* $l \in \lambda + \mathfrak{h}^\perp$ *and* $A \in (U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{il})^{\mathfrak{h}}$, *the action* $\mathrm{d}\tau_l(\overline{A})(\alpha_l)$ *is a multiple of* α_l. *Thus there exists a character* $\lambda_l : (U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{il})^{\mathfrak{h}} \to \mathbb{C}$ *defined by the relation* $\mathrm{d}\tau_l(\overline{A})(\alpha_l) = \overline{\lambda_l(A)}\alpha_l$.

Before we proceed, we need to define two specialization algebras. First we set $H^0_{(\epsilon=1)}(\mathfrak{h}_\lambda^\perp, d^{(\epsilon=1)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}}) := H^0_{(\epsilon)}(\mathfrak{h}_\lambda^\perp, d^{(\epsilon)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}})/\langle \epsilon - 1\rangle$ to be the specialization algebra of the reduction algebra $H^0_{(\epsilon)}(\mathfrak{h}_\lambda^\perp, d^{(\epsilon)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}})$.

Consider a supplementary variable T such that $[T, \mathfrak{g}] = 0$ and set $\mathfrak{g}_T = \mathfrak{g} \oplus \langle T\rangle$ and $\mathfrak{h}_T = \mathfrak{h} \oplus \langle T\rangle$ such that $\dim(\mathfrak{g}_T) = \dim(\mathfrak{g}) + 1$. Set also $U(\mathfrak{g}_T)$ to be the universal enveloping algebra of $\mathfrak{g}_T$ and $U(\mathfrak{g}_T)\mathfrak{h}_\lambda^T$ to be the ideal of $U(\mathfrak{g}_T)$ generated by $\mathfrak{h}_\lambda^T = \langle H + T\lambda(H), H \in \mathfrak{h}\rangle$. Let H be the associated Lie group of $\mathfrak{h}$ and consider the unitary character $\chi_\lambda : H \to \mathbb{C}$ defined by the formula for $Y \in \mathfrak{h}$, $\chi_\lambda(\exp Y) = \exp(i\lambda(Y))$. Denote $C^\infty(G, H, \chi_\lambda)$ the vector space of complex smooth functions θ on G that satisfy the property $\forall h \in H$, $\forall g \in G$, $\theta(gh) = \chi_\lambda^{-1}(h)\theta(g)$. We denote $\mathbb{D}(\mathfrak{g}, \mathfrak{h}, \lambda)$ the algebra of linear differential operators, that leave the space $C^\infty(G, H, \chi_\lambda)$ invariant and commute with the left translation on G.

Recall that from a theorem of Koornwider we have $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}} \simeq \mathbb{D}(\mathfrak{g}, \mathfrak{h}, \lambda)$. Thus setting $\mathbb{D}_T(\mathfrak{g}, \mathfrak{h}, \lambda) := \mathbb{D}(\mathfrak{g}_T, \mathfrak{h}_T, \lambda)$ we can also write $(U(\mathfrak{g}_T)/U(\mathfrak{g}_T)\mathfrak{h}_\lambda^T)^{\mathfrak{h}_T} \simeq \mathbb{D}_T(\mathfrak{g}, \mathfrak{h}, \lambda)$. Finally we define our second specialization algebra $\mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) := (U(\mathfrak{g}_T)/U(\mathfrak{g}_T)\mathfrak{h}_\lambda^T)^{\mathfrak{h}_T}/\langle T - 1\rangle$. In [1, §3.5.3, Theorem 3.5] it is proved, as the outcome of a series of other results that $\mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) \simeq H^0_{(\epsilon=1)}(\mathfrak{h}_\lambda^\perp, d^{(\epsilon=1)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}})$. This result can also be found in [2].

In order to proceed to the character comparison, it is necessary that the theorems of harmonic analysis and deformation quantization refer to the same field. So we need a real character since the whole Kontsevich construction which we mentioned is over $\mathbb{R}$:

Theorem 3. *(See [1, § 4.4.2, Theorem 4.3].) Let* $\mathfrak{g}, \mathfrak{h}, \lambda$ *as before and suppose that the* H*-orbits are lagrangian in the affine space* $\lambda + \mathfrak{h}^\perp$. *Then for a regular* $f \in \lambda + \mathfrak{h}^\perp$ *and such that* $\dim(\mathfrak{h} \cdot f) = \frac{1}{2}\dim(\mathfrak{g} \cdot f)$, *and* $A \in D_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda)$, *the action* $\mathrm{d}\tau_f(A)(\alpha(f))$ *is a multiple of* $\alpha(f)$, *and so there is defined a character* $\lambda^f_{(T=1)} : D_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) \to \mathbb{R}$ *such that* $\mathrm{d}\tau_f(A)(\alpha(f)) = \lambda^f_{(T=1)}(A)\alpha(f)$.

Proof. The idea is to follow the line of proof of Fujiwara proving Theorem 2. This is done by double induction on $\dim(\mathfrak{g})$ and $\dim(\mathfrak{h})$ and works fine up to the case $\mathfrak{h} \subset \mathfrak{g}_0$, $\mathfrak{g}_0$ being a codimension one ideal of $\mathfrak{g}$ (constructed in a standard way using the reduction triplet in the sense of Dixmier). In this case, the condition of Corwin–Greenleaf (see (1) of Eqs. (2.7) in [9]) holds for the character $it\lambda$ and we have

$$\left(U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{it\lambda}\right)^{\mathfrak{h}} = \left(U_{\mathbb{C}}(\mathfrak{g}_0)/U_{\mathbb{C}}(\mathfrak{g}_0)\mathfrak{h}_{it\lambda}\right)^{\mathfrak{h}}. \tag{3}$$

This equation depends rationally on it, $t \in \mathbb{R}^*$. So if (3) holds for it, $t \in \mathbb{R}^*$, it holds also for t in a Zariski-open subset of $\mathbb{R}$ and we write $(U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{t\lambda})^{\mathfrak{h}} = (U_{\mathbb{C}}(\mathfrak{g}_0)/U_{\mathbb{C}}(\mathfrak{g}_0)\mathfrak{h}_{t\lambda})^{\mathfrak{h}}$, but we can't conclude that a similar equation holds for the algebra $(U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$. This is the difference with respect to the proof of Theorem 2 which was about a unitary character. To overcome this setback, we can use polynomial families $t \mapsto u_t \in (U_{\mathbb{C}}(\mathfrak{g}_0)/U_{\mathbb{C}}(\mathfrak{g}_0)\mathfrak{h}_{t\lambda})^{\mathfrak{h}}$. This will allow us to continue the argument and it explains at the same time why the character in the theorem's statement is defined for $D_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda)$ which is elsewhere [1, §3.5.3, Corollary 3.2] shown to correspond to elements who are the value at $t = 1$ of polynomial families $t \mapsto u_t \in (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_{t\lambda})^{\mathfrak{h}}$. The proof then continues by carefully applying the induction arguments on $\dim(\mathfrak{g})$. □

Thus the real character that we construct has a price: The corresponding theorem for the Penney distribution now holds for a smaller algebra: Here becomes also clear the use of the specialization algebra $D_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) \simeq H^0_{(\epsilon=1)}(\mathfrak{h}_\lambda^\perp, d^{(\epsilon=1)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}})$ introduced in [1]. This algebra might be the appropriate object of study when it comes to the Duflo and Corwin–Greenleaf conjectures.

3. Comparison of characters and example

Let $\mathfrak{i}_{(\epsilon=1)}: H^0_{(\epsilon=1)}(\mathfrak{h}_\lambda^\perp, d^{(\epsilon=1)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}}) \hookrightarrow (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$ be the injective map coming from $H^0_{(\epsilon=1)}(\mathfrak{h}_\lambda^\perp, d^{(\epsilon=1)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}}) \hookrightarrow H^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}})$ and $H^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}}) \hookrightarrow (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$.

Theorem 4. *(See [1, §4.4.4, Theorem 4.4].) Let $\mathfrak{g}$ be a nilpotent Lie algebra* $(\dim(\mathfrak{g}) < \infty)$, *$\mathfrak{h} \subset \mathfrak{g}$ a subalgebra, λ a character of $\mathfrak{h}$. Let $P \in H^0_{(\epsilon=1)}(\mathfrak{h}_\lambda^\perp, d^{(\epsilon=1)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}})$ and $u \in D_{(T=1)}(\mathfrak{g},\mathfrak{h},\lambda)$ such that $u = \mathfrak{i}_{(\epsilon=1)}(P)$. Then for a generic $f \in \lambda + \mathfrak{h}^\perp$ there is a pair $(\mathfrak{b}_f, \mathfrak{q}_f)$ satisfying $T_1 \circ \overline{\beta}^{-1}_{\mathfrak{q}_f,(\epsilon)}(P)|_{\epsilon=1}(-f) = \lambda^f_{(T=1)}(u)$.*

Proof. This is done again by a long double induction on $\dim(\mathfrak{g})$ and $\dim(\mathfrak{h})$ confirming that in every step, we compute in the same subspaces for $D_{(T=1)}(\mathfrak{g},\mathfrak{h},\lambda)$ and $H^0_{(\epsilon=1)}(\mathfrak{h}_\lambda^\perp, d^{(\epsilon=1)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}})$ and that the computations match, giving the same character. □

Example. We end the present Note with an example that reveals the power of this approach (the fully detailed and computed example is in §5.5 of [1]): Let $\mathfrak{g}$ be the nilpotent Lie algebra generated by X, U, V, E, Z with relations $[U,V] = E$, $[X,U] = V$, $[X,V] = Z$, $\mathfrak{h} = \mathbb{R}X \oplus \mathbb{R}E$ and $\lambda = E^*$. For a $u \in (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$ and with the right choices (transversal condition) of $\mathfrak{q}_l, \mathfrak{q}$ we have $\beta^{-1}_{\mathfrak{q}_l}(u) = e^{[\frac{1}{12l(Z)}(1-\frac{Z}{2l(Z)})\partial_U^3]}\beta^{-1}_{\mathfrak{q}}(u)$, where $\beta_{\mathfrak{q}}$ is the quotient symmetrization map. If v is a polynomial of $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$ then $\beta^{-1}_{\mathfrak{q}}(v)(l) = e^{-\frac{1}{24l(Z)}\partial_U^3}\beta^{-1}_{\mathfrak{q}_l}(v)(l)$.

The map $\gamma_{CT}: v \mapsto (e^{[\frac{1}{12l(Z)}(1-\frac{Z}{2l(Z)})\partial_U^3]}\beta^{-1}_{\mathfrak{q}}(v))(l)$ is a character of the algebra of differential operators $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$. The important point here is the term in the exponential. This example was long before treated as a counterexample to the idea that the symmterization map β was an algebra isomorphism in this case. Indeed, it was not possible to compute its exact formula for this isomorphism.

The relation $\beta^{-1}_{\mathfrak{q}_l}(u) = e^{[\frac{1}{12l(Z)}(1-\frac{Z}{2l(Z)})\partial_U^3]}\beta^{-1}_{\mathfrak{q}}(u)$ reveals the problem and computes in this case the extra term of third degree with rational coefficients, which can only be computed using the deformation quantization techniques.

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References

[1] P. Batakidis, Déformation par quantification et théorie de Lie, Thèse de Doctorat, Univ. Paris VII, 2009.
[2] P. Batakidis, Deformation quantization and invariant differential operators, C. R. Acad. Sci. Paris, Ser. I 349 (3–4) (2011) 143–148.
[3] Y. Benoist, Analyse harmonique sur les espaces symètriques nilpotents, J. Funct. Anal. 59 (1984) 211–254.
[4] A.S. Cattaneo, G. Felder, Relative formality theorem and quantization of coisotropic submanifolds, Adv. Math. 208 (2) (2007) 521–548.
[5] A.S. Cattaneo, B. Keller, Ch. Torossian, A. Bruguières, Déformation, quantization, théorie de Lie, Collection Panoramas et Synthèse, vol. 20, SMF, 2005.
[6] A.S. Cattaneo, Ch. Torossian, Quantification pour les paires symmétriques et diagrammes de Kontsevich, Ann. Sci. Ecole Norm. Sup. (5) (2008) 787–852.
[7] L.J. Corwin, F.P. Greenleaf, Commutativity of invariant differential operators on nilpotent homogeneous spaces with finite multiplicity, Comm. Pure Appl. Math. 45 (6) (1992) 681–748.
[8] H. Fujiwara, Analyse harmonique pour certaines représentations induites d'un groupe de Lie nilpotent, J. Math. Soc. Japan 50 (3) (1998) 753–766.
[9] H. Fujiwara, G. Lion, B. Magneron, S. Mehdi, A commutativity criterion for certain algebras of invariant differential operators on nilpotent homogeneous spaces, Math. Ann. 327 (3) (2003) 513–544.
[10] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66 (3) (2003) 157–216.