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Hilbert basis of the Lipman semigroup

Base d'Hilbert du semigroupe de Lipman

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ABSTRACT

In this Note, we give a new method to compute the Hilbert basis of the semigroup of certain positive divisors supported on the exceptional divisor of a normal surface singularity. Our approach is purely combinatorial and enables us to avoid the long calculation of the invariants of the ring as it is presented in the work of Altınok and Tosun.

RÉSUMÉ

Dans ce travail, nous donnons une nouvelle méthode pour calculer les générateurs du semigroupe de certains diviseurs positifs a support sur le diviseur exceptionnel d'une singularité de surface normale. Notre approche est purement combinatoire et permet d'éviter le calcul difficile des invariants de l'anneau tel qu'il est présenté dans le travail de Altınok et Tosun.

1. Introduction

The exceptional divisor of a resolution of a singularity of a normal surface is a connected curve. The set of positive divisors supported on this exceptional divisor satisfying some negativity condition forms a semigroup, called the semigroup of Lipman in reference to his work [13]. The unique smallest element of this semigroup characterizes the class of the singularity; for example, if the geometric genus of the smallest element is zero then the singularity is called rational [2]. When the singularity is rational, the elements of the semigroup of Lipman are in one-to-one correspondence with the functions in the local ring at the singularity. These elements are important to understand algebraic and topological structure of the corresponding singularity, see [5,14,15].

The smallest element of the semigroup of Lipman is calculated by the Laufer algorithm (see [12, 4.1]) and all the other elements are computed by the algorithms given in [16,18]. The natural question of determining an explicit finite generating set for the semigroup is answered in [1]. The authors use the tools from toric geometry to compute all the generators by means of the generators of certain ring of invariants. Their method is effective but it is difficult to follow for an exceptional divisor with many components.

Here we present an easier combinatorial method to obtain the set of generators of the semigroup of Lipman. More significantly, we describe another semigroup associated to an exceptional divisor whose Hilbert basis, which can be computed directly from the intersection matrix of the exceptional divisor, gives exactly the generators of the Lipman semigroup and the corresponding ring of invariants at the same time. The latter is important for a deeper study of properties of the associated toric variety, such as being a set-theoretic complete intersection [3] or having a nice Castelnuovo–Mumford regularity [8].

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2. Preliminaries

In this section, we recall some terminology and results which will be used later without any reference. Let Y be a normal surface with an isolated singularity at 0 and $(X, E) \to (Y, 0)$ be a resolution of singularities with an exceptional curve E over 0. Let $E_1, \ldots, E_n$ be the irreducible components of E. The set of divisors supported on E forms a lattice defined by

$$M := \{m_1 E_1 + \cdots + m_n E_n \mid m_i \in \mathbb{Z}\}.$$

There is an additive subsemigroup of M which is referred to as the *Lipman semigroup* and is defined by

$$\mathcal{E} := \{D \in M \mid D \cdot E_i \leqslant 0, \text{ for any } i = 1, \ldots, n\}.$$

It follows that if $m_1 E_1 + \cdots + m_n E_n \in \mathcal{E} \backslash \{0\}$ then $m_i > 0$, for all $i = 1, \ldots, n$, see [2]. By definition, $D \in \mathcal{E}$ if and only if $D \cdot E_i = -d_i$ for some $d_i \in \mathbb{N}$ and for all $i = 1, \ldots, n$. Denote by $M(E)$ the intersection matrix of the exceptional divisor E, that is, a matrix with integral entries defined by the intersection multiplicities $E_i \cdot E_j$. It is known that $M(E)$ is negative definite.

Given $D = m_1 E_1 + \cdots + m_n E_n \in M$, with $m_i > 0$. The following equivalence determines the elements of $\mathcal{E}$

$$D \cdot E_i = -d_i \quad \Leftrightarrow \quad M(E)[m_1 \cdots m_n]^T = -[d_1 \cdots d_n]^T. \tag{1}$$

If $\mathbf{e}_i = [0 \cdots 1 \cdots 0]^T$ is the standard basis element of the space of column matrices of size n, then every column matrix $-[d_1 \cdots d_n]^T$, with all $d_i \geqslant 0$, is spanned by $-\mathbf{e}_1, \ldots, -\mathbf{e}_n$. Hence, it follows that the rational cone over $\mathcal{E}$ is generated by the F_i which is defined to be the (rational) solution of the matrix equation above corresponding to $-\mathbf{e}_i$ for each i. Therefore, we can write F_i as follows:

$$F_i = \sum_{j=1}^{n} \frac{a_{ij}}{b_{ij}} E_j,$$

where a_{ij} and b_{ij} are relatively prime integers. Now, let g_i be the least common factor of $b_{i1}, \ldots, b_{in}$ so that $g_i F_i$ is the smallest multiple of F_i that belongs to $\mathcal{E}$. Denote by M' the lattice generated by $F_1, \ldots, F_n$ and let N, N' be the corresponding dual lattices of M, M' respectively. Then, N' is a sublattice of N of finite index, since M is a sublattice of M'.

Denote by $\check{\sigma}$ the cone in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by the semigroup $\mathcal{E}$. The semigroup $\check{\sigma} \cap M \supseteq \mathcal{E}$ is called the saturation of $\mathcal{E}$ and the semigroup $\mathcal{E}$ itself is called saturated (or normal) if $\check{\sigma} \cap M \subseteq \mathcal{E}$ as well.

Proposition 1. *$\mathcal{E}$ is a pointed, saturated semigroup which is also simplicial and finitely generated.*

Proof. If $D \in \mathcal{E}$, then $D \cdot E_i \leqslant 0$ which forces that $-D \cdot E_i \geqslant 0$. This means that $D \in \mathcal{E} \cap (-\mathcal{E})$ if and only if $D = 0$, which proves that $\mathcal{E}$ is pointed.

Now, take $D \in \check{\sigma} \cap M$, i.e. $D = mD'$, for some $D' \in \mathcal{E}$ and $m > 0$. Since $D' \in \mathcal{E}$, we have $D' \cdot E_i \leqslant 0$ which yields immediately that $D \cdot E_i = mD' \cdot E_i \leqslant 0$. Therefore, D must belong to $\mathcal{E}$ which reveals that $\mathcal{E}$ is saturated.

Since $\mathcal{E}$ is saturated it follows that $\mathcal{E} = \check{\sigma} \cap M$ and thus $\check{\sigma}$ is generated by $n = \dim \check{\sigma} = \operatorname{rank} M$ linearly independent elements $F_1, \ldots, F_n$ over $\mathbb{Q}^+$, which means that $\check{\sigma}$ is a *maximal* and *simplicial* strongly convex rational polyhedral cone. This shows that $\mathcal{E}$ is simplicial.

That $\mathcal{E}$ has a unique finite minimal generating set $\mathcal{H}_{\mathcal{E}}$ over $\mathbb{N}$ follows directly from [17, Lemma 13.1]. $\square$

Definition 1. The unique minimal generating set $\mathcal{H}_{\mathcal{E}}$ of $\mathcal{E}$ over $\mathbb{N}$ is called the *Hilbert basis* of $\mathcal{E}$.

Since $\mathcal{E}$ is saturated, we can associate a normal toric variety $V_{\mathcal{E}} := \operatorname{Spec} \mathbb{C}[\mathcal{E}]$ to $\mathcal{E}$, see [7] for details. It turns out that the coordinate ring $\mathbb{C}[\mathcal{E}]$ of this variety is nothing but the ring of invariants of $\mathbb{C}[M']$ under the natural action of N/N', see [1, Proposition 3.4].

Remark 1. $V_{\mathcal{E}}$ is isomorphic to the geometric quotient $\mathbb{C}^k/G$ in the language of the Geometric Invariant Theory, since $G = N/N'$ is a finite group and $\mathcal{E}$ is simplicial. Hence, $V_{\mathcal{E}}$ has only quotient singularities.

3. Main results

Recall that the unique minimal generating set $\mathcal{H}_S$ of a pointed, saturated semigroup S is called the *Hilbert basis* of S, see [17]. We first associate to $\mathcal{E}$ the obvious subsemigroup of $\mathbb{N}^n$;

$$S_1 := \left\{(m_1, \ldots, m_n) \in \mathbb{N}^n \;\middle|\; m_1 E_1 + \cdots + m_n E_n \in \mathcal{E}\right\}.$$

Proposition 2. *$\phi_1 : \mathcal{E} \to S_1$ is an isomorphism, where $\phi_1(m_1 E_1 + \cdots + m_n E_n) = (m_1, \ldots, m_n)$.*

Similarly, we can associate another subsemigroup S_2 of $\mathbb{N}^n$ with the semigroup $\mathcal{E}$ as follows:

$$S_2 := \{(d_1, \ldots, d_n) \in \mathbb{N}^n \mid d_i = -(D \cdot E_i),\ \text{for some } D \in \mathcal{E} \text{ and for all } i = 1, \ldots, n\}.$$

Proposition 3. *S_2 and $\mathcal{E}$ are isomorphic as semigroups. Moreover, the Hilbert basis of S_2 determines the parametrization of the toric variety $V_{\mathcal{E}}$.*

Proof. Define $\phi_2 : \mathcal{E} \to S_2$ by $\phi_2(D) = (-D \cdot E_1, \ldots, -D \cdot E_n)$, for each $D \in \mathcal{E}$. This defines clearly a homomorphism between the semigroups, since we have

$$(D + D') \cdot E_i = D \cdot E_i + D' \cdot E_i, \quad \text{for any } i = 1, \ldots, n.$$

Surjectivity follows from Eq. (1) together with $M(E)$ being invertible over the rationals. Indeed, for a given $(d_1, \ldots, d_n) \in S_2$ there are non-negative rational numbers m'_i such that $[m'_1 \cdots m'_n]^T = -(M(E))^{-1}[d_1 \cdots d_n]^T$. Multiplying m'_i by the least common factor of the positive integers in the denominators of m'_i, we get non-negative integers m_i such that $\phi_2(D) = (d_1, \ldots, d_n)$, where $D = m_1 E_1 + \cdots + m_n E_n \in \mathcal{E}$. The injectivity follows similarly.

We prove the second part now. Since $\mathbb{C}[\mathcal{E}]$ and $\mathbb{C}[S_2]$ are isomorphic from the first part, V_{S_2} is an embedding of $V_{\mathcal{E}} = \operatorname{Spec} \mathbb{C}[\mathcal{E}]$ in some affine space. It is known that $\mathbb{C}[S_2]$ is generated minimally by the monomials $u_1^{d_1} \cdots u_n^{d_n}$ which is determined by $(d_1, \ldots, d_n) \in \mathcal{H}_{S_2}$. Therefore we need to determine the elements of $\mathcal{H}_{S_2}$ more precisely. Since S_2 is a subsemigroup of $\mathbb{N}^n$ and $\phi_2(g_i F_i) = g_i \mathbf{e}_i$ is the smallest element of S_2 on the i-th ray of the cone $\phi_2(\check{\sigma})$, it follows that H_{S_2} contains $g_i \mathbf{e}_i$, for each $i = 1, \ldots, n$. If we denote by $\mathbf{h}_j = h_{j1}\mathbf{e}_1 + \cdots + h_{jn}\mathbf{e}_n$ the other elements of the Hilbert basis of S_2, then it follows from [10, Corollary 2] that the toric variety V_{S_2} is parametrized by the toric set

$$\Gamma(S_2) = \{(u_1^{g_1}, \ldots, u_n^{g_n}, u_1^{h_{11}} \cdots u_n^{h_{1n}}, \ldots, u_1^{h_{k1}} \cdots u_n^{h_{kn}}) \mid u_1, \ldots, u_n \in \mathbb{C}\}. \qquad \square$$

In order to state our main result, let $A = [M(E)|I_n]$ be the $n \times 2n$ integer matrix obtained by joining the intersection matrix $M(E)$ of the exceptional divisor E and the identity matrix of size $n \times n$. Then, we define the last semigroup as

$$S = \{(v_1, \ldots, v_{2n}) \in \mathbb{N}^{2n} \mid A \cdot [v_1 \cdots v_{2n}]^T = 0\}.$$

Here is the nice relation between the three semigroups defined so far.

Theorem 4. *$S = S_1 \times S_2$.*

Proof. The following observations can be seen immediately.

$$\begin{aligned}(v_1, \ldots, v_{2n}) \in S \quad &\Leftrightarrow \quad A \cdot [v_1 \cdots v_{2n}]^T = 0 \quad \Leftrightarrow \quad M(E) \cdot [v_1 \cdots v_n]^T = -[v_{n+1} \cdots v_{2n}]^T \\ &\Leftrightarrow \quad D = v_1 E_1 + \cdots + v_n E_n \in \mathcal{E} \quad \text{and} \quad D \cdot E_i = -v_{n+i}, \quad \text{for any } i = 1, \ldots, n \\ &\Leftrightarrow \quad (v_1, \ldots, v_n) \in S_1 \quad \text{and} \quad (v_{n+1}, \ldots, v_{2n}) \in S_2.\end{aligned}$$

Therefore, the proof is complete. $\square$

The Hilbert basis of this last semigroup is easy to find and gives important information about the others as we see now.

Corollary 5. *Hilbert basis of S gives the generators of the Lipman semigroup and the parametrization of the corresponding toric variety at the same time.*

Proof. By Theorem 4, it follows that the elements of $\mathcal{H}_S$ is in bijection with the elements of $\mathcal{H}_{S_1}$ and $\mathcal{H}_{S_2}$. Hence, $(m_1, \ldots, m_n, d_1, \ldots, d_n) \in \mathcal{H}_S$ if and only if $(m_1, \ldots, m_n) \in \mathcal{H}_{S_1}$ and $(d_1, \ldots, d_n) \in \mathcal{H}_{S_2}$. Now, it is clear from Proposition 2 that $(m_1, \ldots, m_n) \in \mathcal{H}_{S_1}$ if and only if $m_1 E_1 + \cdots + m_n E_n \in \mathcal{H}_{\mathcal{E}}$. On the other hand, we know from the proof of Proposition 3 that $\mathcal{H}_{S_2}$ determines the parametrization of the toric variety associated to $\mathcal{E}$. $\square$

Remark 2. Our main Theorem 4 gives rise to an algorithm which starts with the intersection matrix $M(E)$ and computes the Hilbert basis $\mathcal{H}_{\mathcal{E}}$ of the Lipman semigroup and the parametrization of the toric variety $V_{\mathcal{E}}$ at once. It uses existing algorithms for computing Hilbert basis of lattice points of cones, where the lattice is given by the kernel of an integral matrix A, see [9] and references therein or [11, Chapter 6].

We conclude the Note with an illustration of our user-friendly combinatorial method.

Example 1. Consider the exceptional divisor E over a singularity of A_2-type. Then $A = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}$.

A computation with a computer package (e.g. CoCoA [4] or 4ti2 [6]) gives the Hilbert basis of S to be the set

$$\mathcal{H}_S = \{(2,1,3,0),(1,1,1,1),(1,2,0,3)\}.$$

This says that $\mathcal{H}_{\mathcal{E}} = \{2E_1 + E_2, E_1 + E_2, E_1 + 2E_2\}$ and the smallest element $E_1 + E_2$ is the fundamental cycle of $\mathcal{E}$. Since $\mathcal{H}_{S_2} = \{(3,0),(1,1),(0,3)\}$, it also says that the corresponding toric variety $V_{\mathcal{E}}$ is parametrized by the toric set $\Gamma(S_2) = \{(u_1^3, u_1u_2, u_2^3) \mid u_1, u_2 \in \mathbb{C}\}$.

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