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Dynamical Systems

A model for the parabolic slices $\mathrm{Per}_1(e^{2\pi ip/q})$ in moduli space of quadratic rational maps

Un modèle pour les sections paraboliques $\mathrm{Per}_1(e^{2\pi ip/q})$ *de l'espace des modules des fractions rationnelles quadratiques*

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A B S T R A C T

The notion of *relatedness loci* in the parabolic slices $\mathrm{Per}_1(e^{2\pi ip/q})$ in moduli space of quadratic rational maps is introduced. They are counterparts of the disconnectedness or escape locus in the slice of quadratic polynomials. A model for these loci is presented, and a strategy of proof of the faithfulness of the model is given.

R É S U M É

Nous introduisons la notion de lieux de parenté dans les sections paraboliques $\mathrm{Per}_1(e^{2\pi ip/q})$ de l'espace des modules des fractions rationnelles quadratiques. Ce sont des analogues du lieu de non-connexité dans la section correspondant aux polynômes quadratiques. Nous présentons un modèle pour ces lieux, et donnons une stratégie de preuve de la fidélité de ce modèle.

1. Introduction

Let $\mathcal{M}_2$ denote the moduli space of Möbius conjugacy classes of quadratic rational maps $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$. Following definitions and statements from Milnor [3], consider loci:

$$\mathrm{Per}_1(\lambda)=\big\{[f]\in\mathcal{M}_2\colon\ f \text{ has a fixed point with eigenvalue } \lambda\big\}\cong\mathbb{C}.$$

Here the focus will be on parabolic slices $\mathrm{Per}_1(\omega)$, with $\omega=e^{2\pi ip/q}$, $p/q\neq 0/1$, i.e. those consisting of equivalence classes of maps with a parabolic fixed point with eigenvalue ω. In such a slice the dynamics is characterized according to the behavior of the critical points. The *relatedness* locus $\mathcal{R}^\omega$ in $\mathrm{Per}_1(\omega)$ is defined by:

$$\mathcal{R}^\omega=\Big\{[f]\in\mathrm{Per}_1(\omega)\colon \lim_{n\to\infty} f^n(c_1)=z_0=\lim_{n\to\infty} f^n(c_2)\Big\},\tag{1}$$

where z_0 is the (persistent) parabolic fixed point and c_1 and c_2 are the critical points of f. The locus $\mathcal{R}^\omega$ is neither open nor closed. It consists of open, connected components of maps where both critical points are in the parabolic basin

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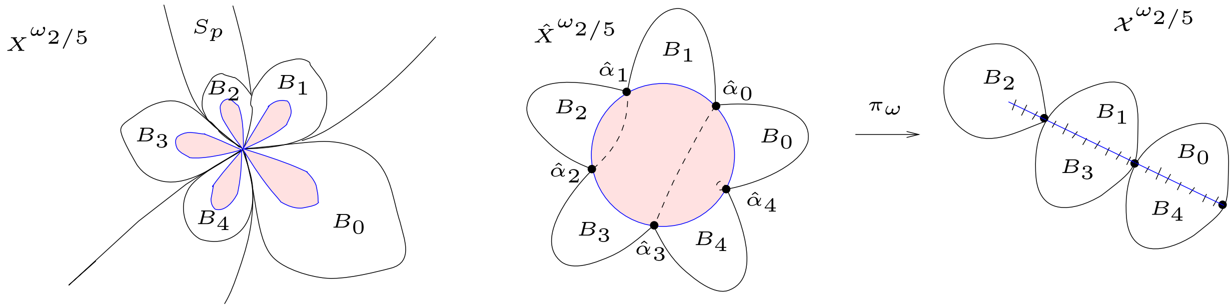

Fig. 1. A sketch of the construction of $\mathcal{X}^\omega$, in the case $p/q = 2/5$.

(in [3] — for hyperbolic components — called *bitransitive* and *capture* components respectively, according to whether both or only one critical point is in the immediate basin, also studied by Rees [6] under different names), a countable set of points corresponding to maps where one critical point is eventually mapped to the parabolic fixed point, and a finite set of points corresponding to maps where the parabolic fixed point is degenerate, i.e. has two q-cycles of components in the immediate basin. In the slice $\mathrm{Per}_1(0)$ the relatedness locus $\mathcal{R}^0$ is the escape locus $\mathbb{C} \setminus M$, where M is the Mandelbrot set, the connectedness locus in the slice of polynomials.

2. The model

The objective is to construct a model for $\mathcal{R}^\omega$ (see Theorem 3.1). Consider the quadratic polynomial

$$P_\omega(z) = \omega z + z^2,$$

with a parabolic fixed point with multiplier ω at 0. This fixed point is called the *α-fixed point*. Let Λ_ω denote the parabolic basin of 0 for P_ω and define also an augmented basin $\tilde{\Lambda}_\omega$:

$$\tilde{\Lambda}_\omega = \left\{ z \in \hat{\mathbb{C}}\colon \lim_{n\to\infty} P_\omega^n(z) = 0 \right\} = \Lambda_\omega \cup \left\{ z \in \hat{\mathbb{C}}\colon \exists n \geqslant 0,\, P_\omega^n(z) = 0 \right\}.$$

The immediate basin has q components, labelled B_j, $j \in \{0, \ldots, q-1\}$ counter-clockwise, so that B_0 contains the critical point $-\omega/2$. It follows from the theory of quadratic polynomials that there are q external rays landing at 0, dividing $\hat{\mathbb{C}}$ into q components. Let S_p denote the component containing the critical value $P_\omega(-\omega/2)$. Let $\phi_\omega : \Lambda_\omega \to \mathbb{C}$ be an extended Fatou coordinate for P_ω^q, i.e. a surjective holomorphic map, of infinite degree, with critical points at the critical point $-\omega/2$ of P_ω and at all its pre-images, so that $\phi_\omega \circ P_\omega^q = 1 + \phi_\omega$.

Normalize ϕ_ω so that $\phi_\omega \circ P_\omega = 1/q + \phi_\omega$ and $\phi_\omega(-\omega/2) = 0$. Let $\mathcal{P}_\delta^j \subset B_j$ be the connected component of $\phi_\omega^{-1}(\{z = x + iy\colon x > \delta\}) \cap B_j$ with the fixed point 0 on its boundary, called a *petal*.

Denote by X^ω the subset of the Riemann sphere obtained by removing the union of the closures of the petals $\mathcal{P}_{1/q}$, $X^\omega = \hat{\mathbb{C}} \setminus \bigcup_{j=0}^{q-1} \overline{\mathcal{P}_{1/q}^j}$. Let $\hat{X}^\omega \cong \hat{\mathbb{C}} \setminus \mathbb{D}$ be the Carathéodory compactification of X^ω, i.e. the disjoint union of X^ω and the set consisting of all prime ends of X^ω. The boundary of $\hat{X}^\omega$ can be naturally identified with the boundaries of the petals, together with q copies of the α-fixed point, corresponding to the q different accesses to the α-fixed point from X^ω. The copies are labelled $\hat{\alpha}_j$, $j \in \{0, \ldots, q-1\}$ counter-clockwise, so that $\hat{\alpha}_j$ is an endpoint of $\partial\mathcal{P}^j$ and $\partial\mathcal{P}^{j+1}$ (Fig. 1). For the remainder of this section sets in $\hat{\mathbb{C}} \setminus \bigcup_{j=0}^{q-1} \mathcal{P}_{1/q}^j$ are to be understood as subsets of $\hat{X}^\omega$, i.e. with q copies of the α-fixed point.

Now an equivalence relation on $\hat{X}^\omega$ is defined. To shorten notation let $\partial\mathcal{P}^j = \partial\mathcal{P}_{1/q}^j$.

Definition 2.1. Two points $z_1 \in \partial\mathcal{P}^j$ and $z_2 \in \partial\mathcal{P}^k$ are called *equivalent modulo p/q*, written $z_1 \sim_{p/q} z_2$, if the following two conditions are satisfied:

- $j + k = 2p \bmod q$, and
- $\phi_\omega(z_1) + \phi_\omega(z_2) = 2/q$.

Two points $\hat{\alpha}_j$ and $\hat{\alpha}_k$ are said to be equivalent modulo p/q if $j + k = (2p - 1) \bmod q$.

Let $\mathcal{X}^\omega$ be the quotient of $\hat{X}^\omega$ under the equivalence relation $\sim_{p/q}$, let $\pi_\omega : \hat{X}^\omega \to \mathcal{X}^\omega$ denote the projection map induced by $\sim_{p/q}$ and let $\nu_\omega = \pi_\omega(\partial\hat{X}^\omega) \subset \mathcal{X}^\omega$ denote the scar after gluing the real-analytic boundaries of the petals back together under $\sim_{p/q}$. It can be proved [7] that the map π_ω gives $\mathcal{X}^\omega$ a Riemann surface structure which extends the initial structure of X^ω, and so that $\mathcal{X}^\omega \cong \hat{\mathbb{C}}$.

Definition 2.2. The model space $\hat{\Lambda}^\omega \subset \mathcal{X}^\omega$ for $\mathcal{R}^\omega$ is defined by $\hat{\Lambda}^\omega = (\tilde{\Lambda}_\omega \setminus (S_p \cup \bigcup_{j=0}^{q-1} \mathcal{P}^j_{1/q}))/_{\sim_{p/q}}$.

The set S_p is removed because it in some sense corresponds to non-realizable matings (i.e. matings of P_ω with maps from the conjugate limb $L_{-p/q}$).

Definition 2.3. From the Fatou coordinate define a tree in $\hat{\Lambda}^\omega$, called a *bubble-tree* and denoted $\hat{\mathcal{T}}^\omega$, by:

$$\hat{\mathcal{T}}^\omega = \pi_\omega\left(\phi_\omega^{-1}(\mathbb{R}) \cup \bigcup_{n>0} P_\omega^{-n}(0)\right) \cup \nu_\omega.$$

The bubble-tree has vertices at pre-fixed and (pre)-critical points of P_ω and at the points $\pi_\omega(\hat{\alpha}_i)$, $i \in \{0, \dots, q-1\}$. A metric is defined on the tree by assigning length one to every edge and letting the distance between any two vertices be the sum of the lengths of the edges in the unique finite path between them.

3. Faithfulness of the model

Theorem 3.1. *There exists a bijective map $\chi^\omega : \mathcal{R}^\omega \to \hat{\Lambda}^\omega$, which is conformal in $\mathrm{int}(\mathcal{R}^\omega)$. The inverse $(\chi^\omega)^{-1}$ is continuous on compact subsets of the bubble-tree $\hat{\mathcal{T}}^\omega$, with respect to the topology induced by the metric on the tree.*

The inverse $(\chi^\omega)^{-1}$ does not extend to $\partial\hat{\Lambda}^\omega$ as an injective map, but it seems tempting to conjecture that it extends to $\partial\hat{\Lambda}^\omega$ as a continuous, surjective map. However, one would expect a proof of continuity to be similar to proving local connectivity of M. A more accessible conjecture would be that $(\chi^\omega)^{-1}$ extends continuously to points in $\partial\hat{\Lambda}^\omega$ that correspond to (pre)-periodic points in $\partial\Lambda_\omega$ (the Julia set for P_ω) and to the boundary of components that correspond to strictly pre-periodic components of Λ_ω.

Let $f_\sigma \in \mathcal{R}^\omega$ be non-degenerate parabolic, let ϕ_σ be a Fatou coordinate for f_σ and let $R \in \mathbb{R}$ be smallest so that the union of petals $\bigcup_{j=0}^{q-1} \mathcal{P}^j_{\sigma,R}$ contains no critical point, but contains (at least) one critical point on the boundary. These petals are called maximal attracting petals and (one of) the critical point(s) on the boundary is called the closest critical point and denoted c_1. The other critical point is then called the second critical point and denoted c_2. The critical values under f_σ are denoted v_1 and v_2 respectively. Normalize the Fatou coordinate ϕ_σ so that $\phi_\sigma(c_1) = 0$ and $\phi_\sigma \circ f_\sigma = 1/q + \phi_\sigma$. Let $\mathcal{U}^0_\omega = \bigcup_{j=0}^{q-1} \mathcal{P}^j_0$ be the maximal attracting petals for P_ω, and $U^0_\sigma = \bigcup_{j=0}^{q-1} \mathcal{P}^j_{\sigma,0}$ the maximal attracting petals for f_σ. Further, let $U^n_\sigma = f_\sigma^{-n}(U^0_\sigma)$ and $\mathcal{U}^n_\omega = P_\omega^{-n}(\mathcal{U}^0_\omega)$. The map χ^ω is constructed via a dynamical conjugacy:

Lemma 3.2. *For all non-degenerate parabolic $f_\sigma \in \mathcal{R}^\omega$ there exists a continuous conjugacy $\eta_{\sigma,\omega} : \overline{U_\sigma} \to \tilde{\Lambda}_\omega$ between f_σ and P_ω, so that $\eta_{\sigma,\omega}(\mathcal{P}^j_{\sigma,0}) = \mathcal{P}^j_0$ for $j \in \{0, \dots, q-1\}$. The domain $U_\sigma = U^n_\sigma$ for some $n \in \mathbb{N} \cup \{0\}$ and $\overline{U_\sigma}$ contains both critical values v_1 and v_2. The conjugacy $\eta_{\sigma,\omega}$ is holomorphic in U_σ.*

Proof. Let $f_\sigma \in \mathcal{R}^\omega$. Recall that ϕ_ω and ϕ_σ are Fatou coordinates for P_ω and f_σ respectively. The map $\eta_{\sigma,\omega} = \phi_\omega^{-1} \circ \phi_\sigma : \overline{U^0_\sigma} \to \overline{\mathcal{U}^0_\omega}$, constructed so that $\eta_{\sigma,\omega}(\mathcal{P}^j_{\sigma,0}) = \mathcal{P}^j_0$ for all $j \in \{0, \dots, q-1\}$, is a homeomorphism, conformal in U^0_σ and it conjugates f_σ to P_ω. If $v_2 \in \overline{U^0_\sigma}$ then $U_\sigma = U^0_\sigma$ and the proof is done. If not, there exists $N > 0$ so that $v_2 \in \overline{U^N_\sigma} \setminus \overline{U^{N-1}_\sigma}$ and the conjugacy extends, by iterated lifting with respect to the dynamics, to a conjugacy $\eta_{\sigma,\omega} : \overline{U^N_\sigma} \to \overline{\mathcal{U}^N_\omega}$. Each lift is chosen to agree with the previous map on their common domain of definition. □

Lemma 3.3. *For all non-degenerate parabolic $f_\sigma \in \mathcal{R}^\omega$, $\eta_{\sigma,\omega}(v_2) \in \tilde{\Lambda}_\omega \setminus S_p$.*

Sketch of Proof. The proof is by contradiction. Assume $\exists f_\sigma \in \mathcal{R}^\omega$ so that $\eta_{\sigma,\omega}(v_2) \in \tilde{\Lambda}_\omega \cap S_p$. Let $\overline{U} = \overline{U^n_\sigma}$ be the maximal domain of the conjugacy $\eta_{\sigma,\omega}$, so that $v_1, v_2 \in \overline{U}$, and $V = \hat{\mathbb{C}} \setminus \overline{U} \cong \mathbb{D}$. Hence f_σ has two univalent inverse branches $f_\sigma^{-1} : V \to V$. Let $\mathcal{P}$ denote the union of q repelling petals at the parabolic fixed point z_0, sufficiently small so that $\mathcal{P} \subset f_\sigma^{-1}(V) \subset V$. If $f_\sigma \in \mathcal{R}^\omega$ so that $\eta_{\sigma,\omega}(v_2) \in \tilde{\Lambda}_\omega \cap S_p$, then $f_\sigma^{-1}(\overline{U})$ separates α from its co-preimage α', and $\mathcal{P}$ is then contained in the image of one of the inverse branches $f_\sigma^{-1} : V \to V$. But then this inverse branch has an attracting q-cycle on the ideal boundary, contradicting the Denjoy–Wolff theorem. □

Strategy of Proof of Theorem 3.1. Definition of the map χ^ω. Let $f_\sigma \in \mathcal{R}^\omega$, with second critical value v_2. If f_σ is degenerate parabolic, then it has two q-cycles of components as immediate basin, with each cycle containing a critical point. In this case choose one of the critical points to be the closest critical point c_1, and name the components in the corresponding cycle

$B_0, \ldots, B_{q-1}$ counter-clockwise, so that B_0 contains the critical point c_1. The components of the other q-cycle will be named counter-clockwise so that the component B'_j is the component between B_j and $B_{(j+1) \bmod q}$. The map $\chi^\omega : \mathcal{R}^\omega \to \hat{\Lambda}^\omega$ is defined by:

$$\chi^\omega(\sigma) = \begin{cases} \pi_\omega \circ \eta_{\sigma,\omega}(v_2) & \text{if } \sigma \text{ non-degenerate,} \\ \pi_\omega(\hat{\alpha}_j) & \text{if } \sigma \text{ degenerate and } v_2 \in B'_j. \end{cases} \tag{2}$$

The map is well defined by Lemmas 3.2 and 3.3, and by the equivalence relation $\sim_{p/q}$, which identifies the images $\eta_{\sigma,\omega}(v_2)$ and $\hat{\alpha}_j$'s respectively, in the cases where there is an ambiguity in the choice of v_2. That the map χ^ω is holomorphic in the interior will follow from holomorphic dependence of the Fatou coordinate ϕ_σ on the parameter σ.

Injectivity follows by a classical pull-back argument, see for example [2] and [5], adapted to the parabolic situation. Surjectivity is proved by constructing a sequence of polynomial-like maps $f_n \in \mathrm{Per}_1(\lambda_n)$, with $\lambda_n \in \mathbb{D}^*$, $\lambda_n \to \omega$ radially, so that the limiting map $f_\sigma \in \mathrm{Per}_1(\omega)$ has the correct position of the second critical value. This is done by using results from [1] on the escape loci in $\mathrm{Per}_1(\lambda)$ and results on convergence of polynomial basins, built upon the star-construction from [4]. Continuity of the map $(\chi^\omega)^{-1}$ on compact subsets of the bubble-tree is proved by using that the map $\eta_{\sigma,\omega}$ preserves the combinatorial structure of the bubble-tree. □

Following Wittner's conjecture on the slice $\mathrm{Per}_2(0)$ [8], and revising a folklore conjecture on parabolic parameter slices, the theorem leads to the conjecture that $\mathrm{Per}_1(e^{2\pi i p/q})$ can be understood as the mating of $\hat{\Lambda}^\omega$ with a truncated Mandelbrot set, $M \setminus L_{-p/q}$, so that the bifurcation locus in $\mathrm{Per}_1(e^{2\pi i p/q})$ is homeomorphic to the mating of $\partial\hat{\Lambda}^\omega$ with $\partial(M \setminus L_{-p/q})$.

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