C. R. Acad. Sci. Paris, Ser. I 348 (2010) 1203–1206

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Differential Geometry

Extension of Reilly formula with applications to eigenvalue estimates for drifting Laplacians ☆

Extension de la formule de Reilly avec applications aux estimées de valeurs propres pour les laplaciens avec dérive

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A R T I C L E   I N F O

Article history:
Received 14 September 2010
Accepted after revision 6 October 2010
Available online 25 October 2010

Presented by the Editorial Board

A B S T R A C T

In this Note, we extend the Reilly formula for drifting Laplacian operator and apply it to study eigenvalue estimate for drifting Laplacian operators on compact Riemannian manifolds' boundary. Our results on eigenvalue estimates extend previous results of Reilly and Choi and Wang.

R É S U M É

Dans cette Note, nous étendons la formule de Reilly au cas des opérateurs Laplaciens avec dérive, et l'appliquons à l'étude d'estimées de valeurs propres pour de tels opérateurs sur des variétés riemanniennes compactes à bord. Nos estimées généralisent des résultats antérieurs de Reilly ainsi que de Choi et Wang.

1. Introduction

Among the important formulae in differential geometry, Reilly formula [5] is an important tool used to give a lower bound of eigenvalues of Laplacian operator on a Riemannian manifold with smooth boundary. Motivated by important work of G. Perelman [19] and the optimal transport theory [22], we study an extension of Reilly formula for drifting Laplacian operator associated with weighted measure and Bakry–Emery–Ricci tensor on a compact Riemannian manifold with smooth boundary. Then we give applications of this formula to the eigenvalue estimates of the drifting Laplacian on manifolds with boundary. The important motivation for such a study is its close connection with fundamental gaps of the classical Laplacian operator on manifolds [16].

Let (M,g) be a compact n-dimensional Riemannian manifold with boundary. Let $L=\Delta$ be the Laplacian operator on the compact Riemannian manifold (M,g). Given h a smooth function on M. We consider the elliptic operator with drifting $L_h=\Delta-\nabla h\nabla$ associated with the weighted volume form $\mathrm{d}m=e^{-h}\,\mathrm{d}v_g$. We also call L_h the h-Laplacian on M. Assume that

$$-L_h u=\lambda u, \tag{1}$$

☆ The research is partially supported by the National Natural Science Foundation of China 10631020 and SRFDP 20090002110019.
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1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.10.003

with the Dirichlet or Neumann boundary condition. We shall always assume that $\lambda > 0$ and $\int u^2\,\mathrm{d}m = 1$. Then $\lambda = \int |\nabla u|^2\,\mathrm{d}m$.

With the help of the Bochner formula for a smooth function f (see [1,9–14,17,18,23,6] and [21]), $\frac{1}{2}L|\nabla f|^2 = |D^2 f|^2 + (\nabla f, \nabla L f) + Ric(\nabla f, \nabla f)$, we can show the following Bochner formula for Bakry–Emery–Ricci tensor (see [2,3,15]):

$$\frac{1}{2}L_h|\nabla f|^2 = \left|D^2 f\right|^2 + (\nabla f, \nabla L_h f) + \left(Ric + D^2 h\right)(\nabla f, \nabla f). \tag{2}$$

We remark that the tensor $Ric^h := Ric + D^2 h$ is called Bakry–Emery–Ricci tensor which arises naturally from the study of Ricci solitons [7].

Then we have $\frac{1}{2}L_h|\nabla u|^2 = |D^2 u|^2 - \lambda|\nabla u|^2 + (Ric + D^2 h)(\nabla u, \nabla u)$. Recall that the second fundamental form of ∂M is defined by $\mathbb{I}(X, Y) = g(\nabla_X \nu, Y)$, where ν is the outer unit normal vector to ∂M. And $H = \operatorname{tr}\mathbb{I}$ is the mean curvature. We shall denote by h_ν the normal derivative of h on ∂M or on the hypersurface P.

Using the integration by part on M, we have the following extension of Reilly formula:

Theorem 1. *We have the following extension of Reilly formula*:

$$\int_M \left(|L_h f|^2 - \left|D^2 f\right|^2\right)\mathrm{d}m = \int_M Ric^h(\nabla f, \nabla f)\,\mathrm{d}m + \int_{\partial M} (H f_\nu - \nabla h \nabla f + \Delta_\partial f) f_\nu\,\mathrm{d}m$$
$$+ \int_{\partial M} \left(\mathbb{I}(\nabla_\partial f, \nabla_\partial f) - \langle \nabla_\partial f, \nabla_\partial f_\nu\rangle\right)\mathrm{d}m. \tag{3}$$

Here and below, the symbol ∇_∂ means covariant derivative taken with respect to the induced metric on ∂M.

We shall apply the above result to study the eigenvalue estimate for drifting Laplacian operators on M. We impose either Dirichlet boundary condition $u = 0$ on ∂M or the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$. The corresponding first nontrivial eigenvalue of the h-Laplacian is denoted by λ_D or λ_N respectively. In below, for notation simplicity, we shall denote by λ for λ_D or λ_N when it is clear in the context.

Theorem 2. *Assume that*

$$Ric + D^2 h \geqslant \left(\frac{|Dh|^2}{nz} + A\right) g \tag{4}$$

for some $A > 0$ and $z > 0$.

(1) *In the Dirichlet case, if the modified mean curvature $H - h_\nu$ of ∂M is non-negative, then $\lambda_D \geqslant \frac{n(z+1)A}{(n(z+1)-1)}$.*

(2) *In the Neumann case, if ∂M is convex, that is, the second fundamental form* (*defined by $\mathbb{I}(X, Y) = g(\nabla_X \nu, Y)$*) *is non-negative, then $\lambda_N \geqslant \frac{n(z+1)A}{(n(z+1)-1)}$.*

Recall that, by definition, a minimal h-hypersurface P in M is a hypersurface P with $H - h_\nu = 0$, where ν is the unit normal vector which defines the second fundamental form of P in M. We denote by Δ_P the Laplacian operator of the induced metric on P. Then we can prove the following result, which generalizes a result of Choi and Wang [4]:

Theorem 3. *Let (M^n, g) be a closed orientable manifold with $Ric^h \geqslant (n-1)K > 0$. Let h be a smooth function on M. Let $P \subset M$ be an embedded minimal h-hypersurface dividing M into two submanifolds M_1 and M_2 (i.e., $H = h_\nu$, this equality being independent on the orientation of the unit normal ν). Then for the drifting Laplacian $\Delta_P^h := \Delta_P - \nabla_P h \nabla_P$, $\lambda_1(-\Delta_P^h) \geqslant \frac{(n-1)K}{2}$.*

This paper is organized as follows. In Section 2 we prove Theorem 1, and Theorem 2 is proved in Section 3. Theorem 3 is proved in Section 4.

2. Proof of Theorem 1

We now prove Theorem 1.

Proof. We shall integrate the formula (2). Choose a set of local orthonormal frame fields $\{e_j\}$ such that $e_n = \nu$ on the boundary ∂M. Note that $\frac{1}{2}\int_M L_h|\nabla f|^2\,\mathrm{d}m = \int_{\partial M} f_i f_{ij}\nu_j\,\mathrm{d}m$, and $\int_M (\nabla f, \nabla L_h f)\,\mathrm{d}m = \int_{\partial M} L_h f f_j \nu_j\,\mathrm{d}m - \int_M |L_h f|^2\,\mathrm{d}m$, where, for the sake of simplicity, we still denote by $\mathrm{d}m$ the measure induced on ∂M.

We shall use the classical notations that $f_i = \mathrm{d}f(e_i)$ and $f_{ij} = D^2 f(e_i, e_j)$, etc. Then we have $\int_M (|L_h f|^2 - |D^2 f|^2)\,\mathrm{d}m = \int_M Ric^h(\nabla f, \nabla f)\,\mathrm{d}m + \int_{\partial M} (f_n L_h f - f_i f_{in})\,\mathrm{d}m$. Recall that $L_h f = \Delta f - \nabla h \nabla f$. Then we have $f_n L_h f - f_i f_{in} = -f_n \nabla h \cdot \nabla f + \sum_{j<n} (f_{jj} f_n - f_j f_{jn})$.

Now

$$\sum_{j<n} f_{jj} = \sum_{j<n} \big(e_j(e_j f) - (\nabla_{e_j} e_j) f\big) = \sum_{j<n} \big(\big(\nabla^{\partial}_{e_j} e_j\big) f - (\nabla_{e_j} e_j) f\big) + \Delta_\partial f$$
$$= H f_n + \Delta_\partial f.$$

For $j < n$,

$$f_{jn} = f_{nj} = e_j(e_n f) - (\nabla_{e_j} e_n) f$$
$$= e_j(f_n) - \sum_{k<n} \mathbb{I}_{jk} f_k.$$

Then we have

$$\sum_{j<n} f_j f_{jn} = \langle \nabla_\partial f, \nabla_\partial f_n \rangle - \mathbb{I}_{jk} f_j f_k.$$

Putting all these together we have

$$\int_M \big(|L_h f|^2 - \big|D^2 f\big|^2\big)\,\mathrm{d}m = \int_M Ric^h(\nabla f, \nabla f)\,\mathrm{d}m + \int_{\partial M} (H f_n - \nabla h \nabla f + \Delta_\partial f) f_n\,\mathrm{d}m$$
$$+ \int_{\partial M} \big(\mathbb{I}(\nabla_\partial f, \nabla_\partial f) - \langle \nabla_\partial f, \nabla_\partial f_\nu \rangle\big)\,\mathrm{d}m.$$

The result follows. □

3. Proof of Theorem 2

The idea in the proof of Theorem 2 is similar to the one used by Reilly in [20] (see also [8]). We use the extension of Reilly formula to prove Theorem 2 below.

Proof. Let $L_h u + \lambda u = 0$. We shall integrate the extension of Reilly formula (3).

Note that $(a+b)^2 \geqslant \frac{a^2}{z+1} - \frac{b^2}{z}$ for any $z > 0$. So, we have $(\Delta u)^2 = (\lambda u + g(\nabla h, \nabla u))^2 \geqslant \frac{\lambda^2 u^2}{z+1} - \frac{|g(\nabla h, \nabla u)|^2}{z}$. Then we have

$$\int_M \big(|L_h u|^2 - \big|D^2 u\big|^2\big)\,\mathrm{d}m \leqslant \int_M \left(\lambda^2 u^2 - \frac{1}{n}(\Delta u)^2\right)\mathrm{d}m \leqslant \int_M \left(\frac{\lambda^2 u^2 (n(z+1)-1)}{n(z+1)} + \frac{|g(\nabla h, \nabla u)|^2}{nz}\right)\mathrm{d}m. \tag{5}$$

Note that for either Dirichlet or Neumann cases, we have

$$\int_{\partial M} \big(H u_\nu - g(\nabla h, \nabla u) + \Delta_\partial u\big) u_\nu\,\mathrm{d}m + \int_{\partial M} \big(\mathbb{I}(\nabla_\partial u, \nabla_\partial u) - \langle \nabla_\partial u, \nabla_\partial u_\nu \rangle\big)\,\mathrm{d}m$$
$$= \int_{\partial M} \big(H u_\nu^2 - h_\nu u_\nu^2\big)\,\mathrm{d}m + \int_{\partial M} \mathbb{I}(\nabla_\partial u, \nabla_\partial u)\,\mathrm{d}m \geqslant 0.$$

In the last inequality we have used our assumption on the geometry of ∂M.

Then by our assumption (4) we have

$$\int_M Ric^h(\nabla u, \nabla u)\,\mathrm{d}m \geqslant \int_M \left(\frac{|Dh|^2}{nz} + A\right)|\nabla u|^2\,\mathrm{d}m. \tag{6}$$

Putting (5) and (6) together we have

$$\int_M \frac{|Dh|^2}{nz}|\nabla u|^2\,\mathrm{d}m + A\lambda \leqslant \frac{\lambda^2(n(z+1)-1)}{n(z+1)} + \int_M \frac{|Dh|^2|\nabla u|^2}{nz}\,\mathrm{d}m,$$

and noting $\lambda \neq 0$, we have $\lambda \geqslant \frac{n(z+1)A}{(n(z+1)-1)}$. The result is proved. □

4. Proof of Theorem 3

Suppose $\Delta_P^h u + \lambda u = 0$. Substituting possibly $-\nu$ to ν, there exists a choice of the orientation of the unit normal vector ν such that $\int_{\partial M_1} \mathbb{I}(\nabla_P u, \nabla_P u)\,\mathrm{d}m \geqslant 0$. Fixing this choice of the orientation of ν between the two open submanifolds M_1 and M_2, we decide to call M_1 the one which admits ν as the unit outer normal vector.

Define f on M_1 such that $L_h f = 0$, on M with the boundary condition $f = u$ on ∂M_1. By Theorem 1 we have $0 \geqslant \int_{M_1}(-|D^2 f|^2)\,\mathrm{d}m \geqslant \int_{M_1} Ric^h(\nabla f, \nabla f)\,\mathrm{d}m + \int_{\partial M_1}(Hf_n - \nabla h \nabla f + \Delta_P u) f_n\,\mathrm{d}m + \int_{\partial M_1}(-\langle \nabla_P f, \nabla_P f_\nu\rangle)\,\mathrm{d}m$. Note that

$$\int_{\partial M_1} (Hf_n - \nabla h \nabla f + \Delta_P u) f_n\,\mathrm{d}m = \int_{\partial M_1} \big((H - h_n) f_n - \nabla_P h \nabla_P f + \Delta_P u\big) f_n\,\mathrm{d}m$$

$$= -\int_{\partial M_1} \big((\nabla_\nu h - H)\nabla_n f + \lambda u\big) f_n\,\mathrm{d}m = -\lambda \int_{\partial M_1} u f_n\,\mathrm{d}m,$$

and

$$\int_{\partial M_1} \big(-\langle \nabla_P f, \nabla_P f_\nu\rangle\big)\,\mathrm{d}m = \int_{\partial M_1} \big(\Delta_P^h u\big) f_n\,\mathrm{d}m = -\lambda \int_{\partial M_1} u f_n\,\mathrm{d}m.$$

Compute

$$2\int_{\partial M_1} u f_n\,\mathrm{d}m = \int_{\partial M_1} \big(f^2\big)_n\,\mathrm{d}m = \int_{M_1} L_h\big(f^2\big)\,\mathrm{d}m = 2\int_{M_1} |\nabla f|^2\,\mathrm{d}m.$$

Using our assumption we have $0 \geqslant ((n-1)K - 2\lambda)\int_{M_1} |\nabla f|^2\,\mathrm{d}m$. Since $\int_{M_1} |\nabla f|^2\,\mathrm{d}m > 0$, we get $\lambda \geqslant \frac{(n-1)K}{2}$.

Acknowledgement

The authors would like to thank the unknown referee for very helpful suggestions.

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